Outline of Superselection Rule on Algebraic Quantum Field Theory : Basic

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Abstract

In physics, the "superselection rule", which can be regarded as an advanced version (strict version) of the "selection rule", is rarely found (in the text of particle theory and quantum field theory). The superselection rule is an important theme in algebraic quantum field theory in relation to the sector theory, the representation theory of the operator algebra (von Neumann algebra) and the theory of irreducible decomposition. Although it depends on the methodology, here I would like to summarize the superselection rule regarding introductory part of the theoretical construction of algebraic quantum field theory.

Keywords: mixed state, pure state, direct sum decomposition, operator algebra, GNS construction

1 Introduction

First, Section 2 gives a general explanation of superposition and mixed states. Then, Section 3 shifts to the viewpoint of the decomposition theorem of Hilbert space theory. Section 4 superselection rule based on the explanation of selection rule and conservation law. Here we also touch on the superselection charge. In Section 5, we consider the basics of the algebraic quantum field theory and the superselection rule through the univalence superselection rule.

Some researchers recognize that the superselection rule is an extremely important theory underlying quantum field theory [17]. The arrangement here is still in the process of development and needs to be updated in the future. I don't have the ability to discuss superselection rule from a single methodology for systematization so far. I'll decline this.

2 Entrance

2.1 General description

If you proceed from a very rudimentary place, it will be difficult to get into the main subject, so I will omit it as appropriate. Many of the rudimentary talks appear in the elementary part of quantum mechanics, so refer to them if necessary. However, the superselection rule is a basic story, but not a rudimentary one.

I don't know the textbook of physics (related to quantum field theory, particle physics) that devoted many pages to mention "superselection rule" except for texts on algebraic quantum field theory and texts related to quantum observation theory). Perhaps this fact is due to the fact that superselection rule is not discussed in standard particle physics lectures at universities and graduate schools[1] Still, if you look at [4] and [6] at hand, Generally, the story is as follows.

The way that the superselection rule is discussed begins with the following definition. That is, consider the inner product created from the linear state of the state vector (Hilbert space vector) that specifies the state of the physical system. Then, the term forming the inner product when each vector is not in the orthogonal state is understood as the "interference term", and it is set as the superposition state. Then, a state that is not so is defined as a mixed state.

Example: when the linear vector combination of two vectors $\psi = \alpha\psi_1 + \beta\psi_2$,($\alpha$, $\beta$is a complex number), the inner product is,

$$\begin{aligned} (\psi, \psi) &= (\alpha\psi_1 + \beta\psi_2, \alpha\psi_1 + \beta\psi_2) \\ &= |\alpha|^2(\psi_1, \psi_1) + |\beta^2|(\psi_2, \psi_2) + \alpha^*\beta(\psi_1, \psi_2) + \beta^*\alpha(\psi_2, \psi_1) \\ &= |\alpha|^2 + |\beta|^2 + \alpha^*\beta(\psi_1, \psi_2) + \beta^*\alpha(\psi_2, \psi_1) \end{aligned} \tag{2.1}$$

Here,ψ_1,$\psi_2$are normalized (norm is 1 : $\sqrt{(\psi, \psi)} = \|\psi\| = 1$) . The last two terms have the structure for causing the interference term.

In (2.1) , when the two vectors are orthogonal, we get $|\alpha|^2 + |\beta|^2 = 1$. Looking only at the vector relationship, there is no superposition anywhere. Since vector orthogonalization is possible by the Gram-Schmidt theorem, the calculation of the inner product above is a closed issue as the geometric structure of Hilbert space [2].

ψ is the object to be collated with the physical observation by taking the "expected value" using the observation A. The expected value is the average value when a certain measurement is repeated. It takes the form $(\psi, A\psi)$ with A as the observation. This comes from Born's stochastic interpretation. Most elementarily, if A has a (discrete) eigenvalue, with $P(a)$ as the expected value is

$$< A >:= (\psi, A\psi) = \Sigma_a P(a), \quad \Sigma_a P(a) = 1 \tag{2.2}$$

When this becomes a continuous eigenvalue (that is, a general spectrum), it becomes an integral using the spectrum measure, which is a generalization of the probability measure[3].

[1]Quantum physics can be divided into finite degree of freedom quantum physics (generally called "quantum mechanics") and infinite degree of freedom quantum physics ("quantum field theory").

[2]In the case of algebraic field theory, the GNS construction method considers operators (representations) or observables first (strictly speaking, observables are not all operators but some classes in them. It is generally self-adjoint operator.) And the general vector is constructed later. Even if it says "after", it is not completely logically "after" because the unit vector is taken as a cyclic vector and a positive linear function on the algebra is considered. However, the method of first setting a linear combination of vectors in Hilbert space and then taking into account operators or observables is not taken in algebraic quantum field theory.

[3]However, if the Hilbert space becomes infinite, the spectral measure does not correspond unless A is self-adjoint operator or unitary operator (or normal operator). In the case of a symmetric operator, it becomes a positive operator valued measure with some conditions relaxed from the spectral measure.

The expected value indicates how much a certain value (in real number) can be obtained by the measurement when a certain physical quantity is measured. Therefore, it is closely related to the so-called observation results (physical phenomena).

So if we add the operator A to (2.1):

$$\begin{aligned}(\psi, A\psi) &= |\alpha|^2(\psi_1, A\psi_2) + |\beta|^2(\psi_1, A\psi_2) + \alpha^*\beta(\psi_1, A\psi_2) + \beta^*\alpha(\psi_2, A\psi_1) \\ &= |\alpha|^2 + |\beta|^2 + \alpha^*\beta(\psi_1, A\psi_2) + \beta^*\alpha(\psi_2, A\psi_1). \end{aligned} \tag{2.3}$$

The last two terms are called interference terms (in quantum physics). However, this interpretation is quite problematic. First, what happens if A is an identity matrix (or unit operator) ? In this case it is same as the inner product of a mere vector, so can it be said that there is an superposition? What happens if A is a projection operator? If $A\psi = Proje_\phi\psi = (\phi, \psi)\phi$ for the vector on which A operates, the expected value is through a simple calculation the absolute square of the transition probability amplitude(ψ, ϕ). It is the absolute square of (ψ, ϕ) and it is written as $|(\phi, \psi)|^2$.This is generally the "transition probability" from ϕ to ψ, but how does this relate to superposition? The simple inner product $\alpha^*\beta(\psi_1, \psi_2) + \beta^*\alpha(\psi_2, \psi_1)$ in the case of the identity matrix is also understood as the amplitude that gives the transition probability. Since it can be done, it can be defined as superposition. After this preface, some commonly used views are: For $\psi = \psi_1 + \psi_2$, if ψ_1 is a proton ψ_2 as a neutron or ψ_1 is 1body electron and ψ_2 is 2 bodies electrons, it is not observed, so it is impossible. The interference term is always 0, and it settles in the state where there is no interference term, that is, the mixed state (classical probability). The non-mixed state is generally regarded as the "pure state", and the pure state is inextricably linked to superposition.

The above view is based on the concept of "gauge invariant". It is also because it has been proved that "the transition probability of protons and neutrons is prohibited" and "the transition of one electron and two electrons is also prohibited".Theoretically, it also means that it cannot be observed because it breaks the gauge invariance. It means that the state of the proton does not suddenly change into a neutron. In addition, it means that there is no state like a mid-proton and a mid-neutron, and if a state of one electron is prepared, it will not be in a state of 1/3 or 1/4 states are not possible).

Sure, we know this fact, but the disappearance of the interference term does not come out from the theory of Hilbert space itself, so the interference term may remain. This point is a issue of the superselection rule.

By the way, many quantum physics texts end with this degree of explanation. Pure states are distinguished from mixed states with interference terms, mixed states result in classical probabilities, and pure states cause superposition and quantum physics-specific phenomena. An example of the latter is the double-slit experiment, which actually involves various difficult problems.

A famous paradox is Schrödinger's cat: An example where ψ_1 is a living cat and ψ_2 is a dead cat. Put the cat in the box containing the radioactive material, and when the radioactive material collapses, the hammer breaks the bottle containing the poison. Put the lid on the box and the life and death of the cat cannot be seen from the outside, and since the decay of radioactive substances is probabilistic, the state of the cat is a superposition of being alive and dead. However, this example is nonsense according to the superselection rule that we will see below. However, what can be said here is that even if there is such a pure state, it is just a discussion of mathematical probability vectors, and it goes into a philosophical discussion as to whether it can be understood by associating it with the existence of cats. For example, it depends on how much the gauge invariant is taken to the minimum, but if the situation where cells are always dying and always reborn is considered so small that it is not "gauge invariant", it is not observed, so it is a meaningless argument.

2.2 Word definition problems

The problem of word definition.: In quantum physics, "interference" is also called "coherence". However, in a broad sense, it seems to mean a state in which various wave waveforms are aligned (geometrical optics, etc.). Attention is required depending on the area of specialization in physics. The opposite is "decoherence", which can also mean that the waveforms do not overlap. However, there are many expressions in physics texts that state vectors (probability vectors) in quantum theory are confused with real waves called "wave functions" (this is largely due to de Broglie's theory). When a vector in Hilbert space is said to be in a coherence state, an image similar to a classical wave may be agitated, but it must be distinguished from quantum theory interference or superposition.

Also, superposition is always discussed together with the pure state of a vector, but it is not conceptually the same, and refers to the case where a vector in a certain Hilbert space cannot be described as a mixed state.

The sum of the pure states is the pure state. The pure state is sometimes combined with the vector state in the theory of operator algebras, which, as [3] points out, is an "Irreducible representation". It is not a very meaningful identification because it is related to how to see. But traditionally, a vector state means that a vector is an end point (this means a pure state), so when different vector states come together, they form a mixed state. This can be the background for the superselection rule, but the superselection rule does not come out directly from the structure of the Hilbert space itself. For example, in the type I von Neumann algebra, generally, the irreducible representation is uniquely determined. Therefore, in finite-dimensional quantum physics, there is only one "sector" that we will see later.

2.3 Product state and Entangled state

I said that since superposition is related to expected value, it is not clear to look only at the geometrical and algebraic structures of vectors. When we include the case where the observable is an identity matrix, we should be able to obtain information about superposition from the structure of the vector. That is the so-called entanglement state and its decomposition ("Schmidt decomposition").

As the simplest example, consider the 2-qubit system that appears in quantum information theory. Even though it is 2 bits, it should be understood here that it means that there are two vector components [8].

For $\psi = (a, b)$, $\phi = (c, d)$, Each component is a complex number, and we calculate the tensor product of the two vectors. Then, it becomes a vector of four components, i.e,

$$\psi \otimes \phi = A = \begin{pmatrix} a\phi \\ b\phi \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

： When it can be written in the product state of the tensor product in this way, it is considered that it is not in the entangled state (untangled state).

on the other hand,for 0=(1,0), 1=(1,0), $\psi = \frac{1}{\sqrt{2}}((0 \otimes 0) + (1 \otimes 1) = \frac{1}{\sqrt{2}}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

This ψ is a unit vector consisting of four components (norm 1), but since it is not multiplied by the product state, this is called the entangled state, and the vectors are said to be superimposed. Therefore, the product state corresponds to the mixed state, otherwise it is the pure state (united state).

Although it is often not explained in detail, a tensor, unlike an inner product, is not a calculation that forms scaler structure like real number, but a so-called vector (similar to-) calculation. In the above case, the 2 components are extended to the vector of 4 components. Therefore, regarding the problem of how it differs from (2.1), the vector whose components are increased by the tensor product is always in a mixed state, we could say? But this is not always true.

This is because the partial system of the pure state may be in the mixed state (to be exact, when a whole pure system can be a partial mixed sate due to the contraction density matrix (operator) on the whole system.) [10]. It is also possible to decompose the entangled state in a certain direction by adding a new degree of freedom. This is called the singular value decomposition [10]. In this case, we must consider the pure state as a state vector given within the size of a Hilbert space (from a mathematical point of view). For example, the middle term of $f(x_1, x_2) = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$ can be not decomposed into two functions. Applying this example to （2.1）, Corresponds to inner product ψ and ϕ. But When we add extra dimensions,

$f(x_1, x_2) = (1, x_1, x_1^2) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix}$, and we can decompose the entangled（one dimension up）.

Therefore, considering the dilation theory of Hilbert space, it must be understood that in mathematics, superposition and mixing are actually situations due to the size of the system.

However, in physics, the interference term is observed, so we have to think about what separates the mixture and the pure state, assuming that it is the same size as the vector considering the Hilbert space. As suggested by the above two-variable function, both states depend on the “base vector”. The mixed state cannot be represented by a single basis vector, and this basis is strongly related to the size of the density matrix.

2.4 Density matrix and state

The explanation of the state by the tensor product is often done in information theory. The explanation of the tensor product representation as a mixed state helps to explain why $\psi = \frac{1}{\sqrt{2}}((0 \otimes 0) + (1 \otimes 1))$ does not become a mixed state and to understand that there are quite a lot of entangled states. Regarding the former, it can be seen that the entangled state cannot be represented by one basis on the Schmidt decomposition, which is an application of the singular value decomposition (when the Schmidt number is 2 or more, it is entangled). Also, the sum of pure states is pure because they have the same basis. The mixed state cannot be represented by the same basis. The mixed state has the sum dimension of two bases. And since the dimension of the Hilbert space formed by the multiplication of two vectors is larger than the dimension of the sum, the tensor product state actually occupies only a part of the Hilbert space (vector space).

As for the Schrödinger cat paradox seen earlier, the life-and-death state of a cat cannot be prepared from the beginning if life and death are orthogonal (mixed state). The paradox is due to the fact that the two vectors represent different states on the same basis. However, as mentioned above, pure and mixed are related to the size of the system, so the so-called "whole system". There may be no reason to keep cats alive and dead in a pure state in the measurement problem for the system.

After all, the pure state corresponds to the special case of the density operator $\Sigma\rho = 1$ according to the Hilbert space theory, which is that the projection operator with the smallest size of a certain Hilbert space is the density operator. Since I don't want to think about the topology issue so far, the Hilbert space is limited to a finite dimension, but when the spatial dimension is d, the decomposition representation is $\rho = \Sigma^l p_k$, $(0 \leq p_k \leq 1$, $\Sigma p_k = 1)$, expected value is becomes as follows; (for $\psi = \Sigma_{m=1}^s \psi_m$)

$$< A >= tr(\rho A) = \Sigma_{i=1}^d (e_i, \psi)(\psi, Ae_i) = \Sigma_{i=1}^d (\psi, Ae_i)(e_i, \psi) = (\psi, A\psi), \qquad (2.4)$$

Of the general expected values (mixed state), the pure state is as follows[4], where one component in the sum of this vector is ψ_1

$$< A >= tr(\rho A) = \Sigma_{i=1}^d (e_i, \psi_1)(\psi_1, Ae_i) = (\psi_1, A\psi_1) \qquad (2.5)$$

In other words, it is the minimum state (end point) of the mixed state.[5]. In this case, the density matrix is a projection to the basis[6].

After all, the binary vector calculated in (2.1) is in a pure state with the same basis. There is no proviso that ψ_1 and ψ_2 are mixed states of a certain ratio (2: 3, etc.) and have different bases, but the mixed states are not the same regardless of any conversion of the bases. So, if ψ_1 and ψ_2 are orthogonal, no superposition occurs (this forms the meaning of the superselection sector).

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[4]For the third equation, it is calculated in the Dirac representation or Schatten formal of $\Sigma_k p_k |\psi_k >< \psi_k| = \Sigma_k(\psi_k,)\psi_k$, which is a series that converges with respect to the trace (in the case of infinite dimensions, with respect to the trace norm). This is because the trace is the sum of the inner products of the operators on the orthogonal basis..

[5]$p_1 = 1$,which is $\rho = |\psi_1 >< \psi_1|$ in Dirac notation.

[6]Density operators have some definitions and theorems as operator theories, but they are not discussed here.

This area is not often mentioned in mathematical texts. Superposition occurs for vectors whose bases cannot be separated discretely due to Gram-Schmidt orthogonalization. Therefore, "superposition of protons and neutrons does not occur" or " when arguing that the superposition of live and dead cats is ridiculous", before heading to the philosophical debate of reality, It doesn't make sense if you don't show what really the basis transformation is or what the basis representations. Unless we discuss whether or not there is a projection that divides a vector into orthogonal states (if it does not exist, what is its meaning), we cannot understand anything by looking only at the vector in Hilbert space.

As an example, we take the two-component vector $\psi = (1,0)^t$ and $\phi = (1,0)^t$ as an orthonormal basis. There are several cases.

1) Whenψ and ϕ is equal ratio mixing: Eachψ,ϕ has different base. The density matrix is $\rho = \frac{1}{2}|\psi><\psi| + \frac{1}{2}|\phi>|<\phi| = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}I_2$

2)when $\psi$and ϕ are pure states: .Each base is same, so When the vector Ψ composed of ψ, ϕ, it becomes $\Psi = \frac{\psi+\phi}{\sqrt{2}}$ and the density matrix is $\rho = |\Psi><\Psi| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

2') In case of creating a pure state different from the pure state of 2): Let the vector be Φ so$\Phi = \frac{\psi-\phi}{\sqrt{2}}$is and the density matrix is $\rho = |\Phi><\Phi| = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

3) A new mixed state can be created by mixing the vectors created in 2) and 2') as in 1).

4) If 2) and 2') are changed to $\Pi = \frac{\Psi+\Phi}{\sqrt{2}}$ as in 2), a pure state can be created.

In this way, how to take the basis determines the mixed state and the pure state. The fact that the basis is the same means that, from the standpoint of mathematics, the components on it are transformed all at once, as is clear from linear algebra (in operator theory, the operators are transformed all at once). In the mixed state, each basis is separate, so in (2.1), the ψpart and the ϕ part are not linked in the basis.

The calculation $(a+b) = a^2 + 2ab + b^2$ is a component calculation on the same basis (when viewed as $(a+b)e_1$). When we consider that $ae_1 + be_2$, which has two orthogonal bases, is squared, $2ab$ does not remain (0 because it is orthogonal). $(a+b) = a^2+2ab+b^2$.

When discussing Schrödinger's cat, as a physical phenomenon, "a cat vector" that lives and dies on the same basis must be prepared from the beginning. Aside from the profound (?) Philosophical definition of what it means to be alive or dead, such a state is not prepared by ordinary phenomena. I've never seen such a cat (I've seen a dying cat, but it's still alive). Therefore, Schrödinger's cat paradox is, in a sense, barren. By the way, let's imagine in my head what the cat on the other side of the earth is like (imagination in the sense of observing). However, the probability is just a mixed probability, whether it is alive or dead. So the state to be prepared is not a pure state. This issue is also considered in relation to the problem of observation. If there is life and death on the same basis, life and death are not primary independent, so if life is multiplied by a constant (whether it is extended or reduced), it will die halfway. What determines the length of a constant multiple? It doesn't make sense if you don't know it. However, there can be a pure state in the microscopic object of quantum physics. This is because orthogonal (contradictory) concepts such as life and death do not dominate the quantum world. One example is the “wave and particle” picture of a micro object, which is cited in the Copenhagen interpretation. This ordinary language expression can be misleading. The wave picture and the particle picture are not orthogonal in quantum theory, but when they are expressed as “AND”, there is a possibility that they will be reduced to classical mechanical expressions. Rather, it should be expressed that a micro object has BOTH particle and wave properties.

I will touch on the double slit experiment on another opportunity.

3 Explanation by orthogonal space

Considering the classifications from 1) to 4) in the previous section, the Hilbert space in which the state exists is classified. This is systematically discussed in [9] and [11].

The issue here is related to the direct sum decomposition of Hilbert space in mathematics. The direct sum decomposition is represented by the discrete case and the integral (including the continuous decomposition). However, the case where the sector of superselection rule to be considered later are continuous has various difficult aspects in terms of physics, so for the time being, discrete decomposition is the main focus.

The direct sum decomposition of a Hilbert space ($\mathcal{H}$)means decomposes into closed subspaces that are orthogonal to each other ($\mathcal{H}_n$, $n \in N$) [1], (N: natural number) .

This is generally as follows:

$$\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n \tag{3.1}$$

Direct integrals require the conditions of measure theory.[7]

Now, (3.1) is a discussion of mathematics, but here we reconfirm the meaning of direct sum decomposition. The point is that the vectors belonging to each closed subspace are orthogonal to each other. This means that if we take an arbitrary vector from different closed subspaces and calculate the inner product, it will be 0, as we saw in the previous section. This orthogonal decomposition is an important background of the superselection rule.

The Japanese translation of Bogolubov’s text [9] is considerably more compact than the later published text [13] in English. What is discussed in [9] is that the Hilbert space in the quantum system state where the superselection rule (mixed state) works is formulated as a direct sum of closed subspaces where the superselection rule does not work(pure state: superposition). When the Hilbert space is directly decomposed, each closed subspace is called a coherent space, and each coherent space belongs to the equivalence class in the pure state, and superposition (coherent) does not occur between different subspaces. Although the details are not discussed much in the Japanese translation text, the projection operator is defined as the mercumal that separates each subspace.

When operator A makes any closed subspace$\mathcal{H}_\alpha$of $\mathcal{H}$ invariant or when $\mathcal{H}_\alpha$ is nontrivial invariant subspace of A, $A$is an “Observable” (in some$\mathcal{H}_\alpha$) .α can be infinite

[7]For example,$L^2(\Omega, d\mu, \mathcal{H}) = \mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$, Ω is a set of real number and for each$\omega \in \Omega$, $\psi(\omega)$ is vector value function on $\mathcal{H}$. $\mathcal{H}_\omega$is regarded as Fiber in this Integral[2].

$(\alpha \in N)$.

Therefore, let p_α be the projection operator for each of the elements Ψ_α of $\mathcal{H}$, and the sum be$P = \Sigma_\alpha x_\alpha p_\alpha$ (where x can be an observable itself if it is a real number, but it can also be a complex number). This P classifies the entire Hilbert space into some coherent subspaces. Then, A is also decomposed into the diagonal sum of $A = \Sigma_\alpha p_\alpha A_\alpha p_\alpha$ and the self adjoint operator. Naturally, $[A, P] = 0$, and the same exchange relationship holds in each subspace.

Furthermore, since this direct sum decomposition is a decomposition into a coherent subspace, that is, a pure state, the operators in commutation relation with each A_α are constant multiples of the unit operator. This means that the space has been irreducibly decomposed, and if the observable is of typeI von Neumann algebra, it means that the irreducible decomposition of A is unique[8]. Therefore, as shown in [14], in the case of typeI, the irreducible representation (coherent subspace) becomes the whole (all become the same), so the superposition is mainly played by the typeI von Neumann algebra. For example, typeⅡ andtypeⅢ type do not appear in superposition in the operator algebras due to the non-uniqueness of the irreducible decomposition and the nature of the projection derived from the direct integral decomposition theory).

The projections mentioned above play the role of dividing the Hilbert space into direct sums, but when this is physically seen, they play the role of superselection charges. The vectors of each closed subspace are No superposition occurs. That is, a classical probability is provided. It also defines the nature of the observed quantity.

4 From selection rule to superselection rule (explanation of conservation law)

Before formulating the superselection rule, let's look at the physical meaning. The superselection rule has "super" attached to it, which is a further (mathematical) generalization (or specialization depending on the viewpoint) of the physical rule called "selection rule". Few Japanese physics texts describe this selection rule, and [5] has only a few.

The selection rule seems to have various definitions, but in general, it is related to the existence of conserved quantities. The existence of conserved quantities such as Hamiltonian, which represents energy, is a strong limitation on the time evolution (time shift) of the system. This limitation is called the "selection rule". For example, the selection rule works when there is no process that breaks the spatial inversion invariance (parity invariance) before and after the scattering and decay of elementary particles. Or, when an atom emits one photon, the angular momentum of the atom is limited to 0 or ± 1, and there is no transition that becomes ± 2.

This means that when a physical quantity is X, the value of X does not change even with time evolution. Ignoring the mathematical rigor, it means that the conserved quantity X can be written with Hamiltonian as H, $HXH^* = X$, or $[X, H] = 0$[9].

[8] From the standpoint of sector sector theory in algebraic quantum field theory, it is the same as having only one sector. Quantum physics with finite degrees of freedom is described by typeI, so the sector is one [3].

[9] H^* represents conjugation, $[A, B] = AB - BA$ represents CCR (canonical commutation relation).

The superselection rule, according to [7], is formulated as a rule that the value of this X "does not change with any observable operation". That is, it generalizes the Hamiltonian above (C). We regard it as a superselection charge and require that C be commutative with all measurable amounts X ($[C, X] = 0$). In this case, X must be a self-adjoint operator, and C is often also a self-adjoint operator (as can be seen in the discussion of direct sum decomposition above, a projection operator is a self-adjoint operator). It means that "If a certain physical quantity is measurable, the physical quantity satisfies the superselection rule". According to [7], the action of the superselection rule is formulated as a "necessary condition" for a self-adjoint operator to become a measurable physical quantity. It seems that the sufficient condition is unknown, but if the even number is taken, if $[X, C] \neq 0$, then X is not a measurable physical quantity (even if it is a self-adjoint).

Therefore, from a mathematical point of view, the superselection rule also plays a role in selecting measurable observables from all self-adjoint operators. There are several types of superselective charges above, which also divide the types of superselection rules [17].

Then, the Hilbert space (or Fock space) that describes the space of the quantum field is directly sum-decomposed by a certain kind of superselection charge (mixed state). The sector theory in algebraic quantum field theory treats the decomposed subspace (pure state) as a sector sector and treats the vacuum and the excited state from it.

At last, the preparations for the algebraic quantum field theory are halfway.

5 Algebraic quantum field theory

Algebraic quantum field theory considers (local) observables first, and derives vectors in Hilbert space in representations of observables. Therefore, it is a very important method for considering the observation problem, and since the quantity of observation is closely related to the superselection rule as seen above, the superselection rule is naturally an important theory. Here, among the classes of operator algebras, the von Neumann algebra is mainly organized. As literature, [11], [12], [15], [16], and [17] are classical and important.

5.1 Representation of operators

The representation of operator $\pi(A)$ of the physical quantity A is generally defined as the $*$ homomorphism from the C^* algebra containing the von Neumann algebra. More precisely, $\mathcal{A}$ is the C^* algebra, $\mathcal{H}$ is the Hilbert space, and defined as $*$ homomorphism from $\mathcal{A}$ to the bounded operator $\mathcal{B}(\mathcal{H})$. Based on this representation a positive linear functional as "state ψ" [10] is defined.

The most basic of states is the GNS construction method, which is the theorem that "states are vector states (pure states) of appropriate representation".

[10] I will omit this area. See [11], [15], etc. as appropriate.

Theorem 5.1

For any state ψ over C^*algebra$\mathcal{A}$,there exist a Hilbert space $\mathcal{H}_\psi$, a representation $\pi_\psi$of $\mathcal{A}$,and unit vector Ω_ψ, satisfying the following.
(1) For any $A \in \mathcal{A}$, $\psi(A) = (\Omega_\psi, \pi_\psi(A)\Omega_\psi)$
(2) Ω_ψ is cyclic vector of π_ψ,
i.e., $\pi_\psi(\mathcal{A}) := \{\pi_\psi(\mathcal{A})_\psi | \mathcal{A} \in \mathcal{A}\}$ is dense in $\mathcal{H}_\psi$.
If these two conditions are satisfied, $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ is unique as a unitary equivalence class.

In the algebraic quantum field theory, instead of considering the superposition of Hilbert spaces themselves, discussions are made using this state. In the GNS construction method, it is difficult to see the general vector of Hilbert space, but in fact, the general vector composition is hidden in the representation (and the unit vector), and it is not an outlandish calculation.

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5.2 The most basic explanation (from Araki's text): Univalence superselection rule

In the famous text of algebraic quantum field theory by Professor Huzihiro Araki [11], the superselection rule is first taken up as a historically well-known univalence superselection rule.This univalence is also called the selection rule by single-valueded representation. This is the superselection rule proposed by Wick, Wightman, and Wigner. As explained in [13], it is based on the representation theory related to the spinor field.

The unitary representation of $(a, A) \in \tilde{\mathcal{P}}_+^\uparrow \to U(a, A)$[11] of the universal covering $\tilde{\mathcal{P}}_+^\uparrow$ of a class of Poincaré group[12]$\mathcal{P}_+^\uparrow$ is always giving $U(a, -I) = uU(a, I)$[13].There are cases where u is single-valued or monovalent ($u = I$) and double-valued or divalent ($u = \pm I$). The spinor field corresponding to the double-valued representation of the Poincaré group cannot be observed, because the superselection rule cannot transform two different subspaces. However, it is possible if they are unitary equivalence, but in the double valued representation, the conversion from one to the other is discretely separated by the different landmarks I and$-I$. Therefore, they are not equivalent to each other, that is, the spinor field of the double-valued representation can not be observable. However, in the case of single-valued, the Hilbert space is separated into each subspace. Therefore, the representation spaces are determined according to each univerlence u[14].

Here is the basic idea of the superselection rule. The superselection rule confirms that the fields that follow the Fermi statistics, such as the spinor field and Dirac field above, are not directly observable. If u is regarded as an selection charge, a field (state) that causes a transformation connecting different coherent subspaces is unobservable. Therefore, it

[11] where a is the element ofR^4 (four-dimensional real number) indicating translation, A is Lorentz transformation, and the element of $SU(2, C)$ (C is a complex number).

[12] Also known as the restricted inhomogeneous Lorentz group.

[13] I is unit vector

[14] u is written concretely, for example, when the angular momentum L_z is in the z direction, it is $u = e^{2\pi i L_z/\hbar}$, which gives a 360-degree rotation.

can be seen from this that not all self-adjoint operators or self-adjoint representations are observable. A half-odd number of spins (a number expressed by $n + 1/2$ with n as an integer. An electron is 1/2) is a double-valued representation, so it is unobservable.

Returning to the previous unitary representation, it is $U(a, -I) = uU(a, I)$, which is $u = \pm I$[15], $U(a, -I)$ gives a double-valued representation.

The following is a description of what is in Araki's text.

For the translation of $U(a, I) = e^{-iPa}$, $u^2 = I$. By $A \in SU(2, C)$, u does not return by 360 degree rotation, but returns by 4π rotation[16]. The physical quantity and state do not change when returning 360 degrees (rotation). If the representation is $u = \pm 1$, it will be $\pm U(a, A)$, and $\pi(A) = \Lambda$, (Λ will indicate Lorentz transformation).

Therefore, the problem is that it should be shown that the superposition state is not realized in the case of double-valued representation. This is because if u, which takes two different values, belongs to the same subspace, the superposition works. However, this is not the case, and it is in a mixed state.

Since representation $\pi(A)$ of physical quantity of A and and u are clearly commutative, it means that, for a unit vector $u\Psi_1 = \Psi_1, u\Psi_2 = -\Psi_2$, two (GNS) representations $\omega_1(A) = (\Psi_1, \pi(A)\Psi_1)$, $\omega_2(A) = (\Psi_2, \pi(A)\Psi_2)$ belong to another subspace.

So,for the vector $\Psi = \alpha\Psi_1+\beta\Psi_2$,$(0 < |\alpha|, |\beta| < 0, |\alpha|^2+|\beta|^2 = 1)$, u is a projection and separates the two vectors. As a result $\omega(A) = (\Psi, \pi(A)\Psi)$ becomes the mixed state $\omega = |\alpha|^2\omega_1 + |\beta|^2\omega_2$. At this time, the mark (u) that is commutative with the representation $\pi(A)$ of the physical quantity is generally called the superselection rule (superselection charge), and the Hilbert space is directly sum-decomposed and its coherent subspace (equal class) and plays a role of sorting into coherent subspaces (equivalence class spaces)).

Theorem 5.2

In order for the two states ω_1, ω_2 to be non-superposition, a intertwitting mapping (includes unitary mapping)[a] which connects the GNS representations that accompany each π_{ω_1}, π_{ω_2} is only 0 (this is called "disjoint" representation).

[a]In the case of irreducible representation, unitary non-equivalence is meaningful, but consider other cases as well. See [3] for this.

On the contrary, if the two representations are not disjointness, the orthogonal projections do not accompany the representations and whole space is not be separated into different subspaces.

If a representation $\pi(A)$ is an irreducible representation, then its commutative algebra is only a constant multiple of the complex number, so u that satisfies $[u, \pi(A)] = 0$ is $CI = \pi(A)'$, which represents superposition (coherent). For the single-valued representation $U(a, I)$ of the Poincarè group mentioned above is, so the superselection rule (charge) is $u = I$.

[15]This formula is a unitary operator and an anti-unitary operator. And, physically, it is derived from the relation of time inversion and space inversion in the Poincaré group that accompanies it.

[16]Locally isomorphic to $SO(3)$.

5.3 Note: Physical consideration of superposition

By the way, it seems that the story is not so easy when it comes to whether the physical state of this univalence superselection I is realized in a vacuum state. Hilbert space theory and representation theory do not appear in the general theory of quantum field theory as physics. Excitation from a vacuum is possible with a creation operator, but the vacuum state and the excited state are different in the first place. Even in the above example, particles with spin 1/2 cannot be found in the representation space of the vacuum state, because the excited state is different from that of the vacuum (slightly different from the divalent problem of the spinor field)[17].

In the scattering problem, scattering is described by the asymptotic field. In the process of creating an excited state for a vacuum vector using creation (and annihilation) operators, the representation space $\mathcal{H}_\omega$ of vacuum state ω and representation space of a relativistic one-particle state with spin 0, mass m are different. A vacuum state and a state in which one particle is moving in the scattering process (moving state under a force field) $\mathcal{H}$ and a space $\mathcal{H}_0$ in which it is moving freely (linearly) in an approximate field must be separated. In the theory called "Haag-Ruelle's scattering formula", first, the scattering operator is identified as $\mathcal{H} \sim \mathcal{H}_0$[18]. And superposition of particles with the same class of spin and mass is allowed[19].

Furthermore, the asymptotic behavior of the particles is constructed as a unitary map from $\mathcal{H}$ to $\mathcal{H}_\omega$. Therefore, in this scattering theory, a state space with particles is assumed as a subspace in a vacuum state. For example, the one-particle state and the vacuum state can be superposed. However, it should be noted here. Although it does not appear mathematically, not all particles can be superposed on a vacuum. Vacuums and bosons (eg., photons) are possible, but fermion (eg., electrons) and vacuums are not. So, the reason why the superselection rule does not come to the fore in Haag-Ruelle's scattering formula is that it is limited to particles that can be superposed from the beginning.

Therefore, the "sector theory" includes the case where superposition is possible as a special case, and the relationship between the vacuum and the disjoint representation space and the other disjoint spaces are organized in order from the vacuum sector. Sector theory takes into account the relationship between each representation spaces From a physical point of view, sector theory is a categorization of the world underlying physical phenomena, and shows us that the Nature-World is actually shaped by a great variety of aspects. However, this view seems to be the opposite from the viewpoint of modern physics, where the monism of nature is the highest goal. But at least from the typeIII theory of the von Neumann algebra, which plays an important role in sector theory, what kind of meaning does monism (as like String Theory) have? We may also consider this meaning philosophically.

[17]In the spinor field ψ,the double-valued representation of 360 degree rotation $u^*\psi u = 1$, $u^*\psi^* u = -\psi^*$ is unobservable. However, the single-valued representation $\psi^*\psi$ etc. could be observed. However, this $\psi^*\psi$ is the product of two Fermi, not an odd number.The product of one Fermi field or an odd number of Fermi fields is half-integer, so it seems unobservable (?): This is currently unknown to me.

[18]Mathematically, concerning the scattering operator the main issue is the completeness of the wave operator and the unitary problem of the scattering operator.

[19]This is a classification into a contracted representation based on the covering representation of the Poancarè group.

Finally, we interpret Schrödinger's cat from the viewpoint of superselection rule. The work of observation is a macro (classical physical) work. In that case, the classical probability in the mixed state works in the observation scene. Given that we can have a superposition state that allows the cat to be alive or dead at the same time, if we apply the univalence superselection rule, we prepare an representation of irreducible state and we have to consider whether this representation can be superposed with the vacuum state. Or we have to consider what the superselection charge $u = CI$ means. There seems to be a counterargument that the superposition state is at the level of radioactive substances, and cats are macroscopic objects. If so, it is meaningless unless you specify where to measure the difference between a cat and a micro-radioactive element such as radium. If a cat is also a quantum object, (living cat vector) + (dead cat vector) should be considered according to the finite particle state. If so, we can fully understand it using the factor theory of the type I von Neumann algebra. So that means that the representation of the physical quantity that measures this cat is closed in an unobservable micro system. Because the classical world (observation site) is a mixed world where the superposition state disappears [3]. Therefore, in any case, Schrödinger's cat paradox becomes a barren phantom problem.

This is the end of this time. If there is an opportunity, I plan to further sort out the following problems.

・ Superselection charge and sector
・ Pure state in product state
・ Double slit problem
・ Split property
・ issues of time

, etc.

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