Numerical 3-D Shape Inference from Shading with New Type of Constraint

Tadayoshi SHIOYAMA  Wen Biao JIANG
Department of Mechanical and System Engineering
Kyoto Institute of Technology
Matsugasaki, Sakyō-ku
Kyoto 606, Japan

Abstract

In traditional three dimensional (3-D) shape inference from shading, surface normal distribution is estimated by using a traditional constraint imposed on surface normal and image brightness. In this paper, we derive another new type of constraint imposed on surface normal and gradient of image brightness, and propose 3-D shape inference algorithm by using both the traditional constraint and the new type of constraint. We prove that incorporation of the new type of constraint in an algorithm for recovering surface normal distributions from image brightness speeds convergence as compared to a similar algorithm that does not employ this constraint. The usefulness of the new type of constraint is shown by numerical experiments.

1 Introduction

Since Horn[1] has shown that image brightness is related to the surface normal of 3-D object, the recovery of shape from shading has been extensively researched[2]. Ikeuchi and Horn[3] have proposed an algorithm for estimating 3-D shape by using the traditional constraint imposed on surface normal and image brightness. Woodham[4] has investigated the relationship between the gradient of image brightness and surface curvature. In this paper, we derive another new type of constraint imposed on surface normal and gradient of image brightness on the basis of differential geometry, and propose 3-D shape inference algorithm by using both the traditional type of constraint and the new type of constraint. We theoretically consider the effect of the new type of constraint on estimation of surface normal distribution. We prove that the rate of convergence of the algorithm becomes faster by using the new type of constraint in addition to the traditional constraint. Furthermore, the usefulness of the new type of constraint is shown by numerical experiments.

2 Algorithm for 3-D Shape Inference

We assume orthographic image projection and let the viewing direction be parallel to the z-axis. Then, the 3-D shape of an object can be described by its height, z, at coordinate (x,y) in image plane. We denote by n a unit vector normal to the surface of the object, and by s a unit vector aligned with the direction of the light. The vector n and s are described by points on the unit sphere called Gaussian sphere. In the stereographic projection, a point on the Gaussian sphere is projected by a ray through the point from the south pole onto the tangent plane at the north pole, which is called the stereographic plane. The coordinate (f,g) in the stereographic plane are given as follows:

\[ f = 2p[\sqrt{1 + p^2 + q^2} - 1]/(p^2 + q^2) \]  
and \[ g = 2q[\sqrt{1 + p^2 + q^2} - 1]/(p^2 + q^2), \]

where p and q are defined as

\[ p \equiv \frac{\partial z}{\partial x} \text{ and } q \equiv \frac{\partial z}{\partial y}, \]

and are related to f and g as follows:

\[ p = 4f/(4 - f^2 - g^2) \text{ and } q = 4g/(4 - f^2 - g^2). \]

Since the surface normal \( \mathbf{n} \) is given by

\[ \mathbf{n} = (-p, -q, 1)/\sqrt{1 + p^2 + q^2}, \]

the vectors \( \mathbf{n} \) and \( \mathbf{s} \) are described in terms of \( f \) and \( g \) as follows:

\[ \mathbf{n} = [-4f, -4g, 4 - f^2 - g^2]/(4 + f^2 + g^2) \]  
and \( \mathbf{s} = [-4f, -4g, 4 - f^2 - g^2]/(4 + f^2 + g^2). \)

where \((f_s, g_s)\) denote the stereographic coordinate corresponding to the direction of the light.
We assume that the viewing direction coincides with the north pole of the Gaussian sphere and assume that points on the northern hemisphere of the Gaussian
sphere are considered. Therefore, the considered points \((f, g)\) and \((f_j, g_j)\) in the stereographic plane are constrained to the following regions: \(f^2 + g^2 \leq 4\), \(f_j^2 + g_j^2 \leq 4\). We deal with a surface material which exhibits Lambertian\(\text{[3]}\) reflection. Then, the reflectance map is given by \(n = \text{max}[0, n \cdot s]\) where

\[
R(f, g) = \frac{16(f, f + g)}{(f + f^2 + g^2)(4 + f^2 + g^2)} \equiv R(f, g),
\]

and the symbol \(\cdot\) denotes a scalar product. We denote by \(R(f, g)\) the right-hand side of the equation of \(n \cdot s\). When we denote by \(E\) the brightness normalized such that its maximum value is equal to 1, we obtain the traditional constraint imposed on the surface normal and brightness:

\[
E = R(f, g) = n \cdot s.
\]

2.1 Constraint imposed on the gradient of brightness

In this section, we derive the new type of constraint which is imposed on the gradient of image brightness. When we describe a surface in 3-D space in terms of parameters \(u^1\) and \(u^2\) as \(r(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2))\), the vectors defined as

\[
r_1 \equiv \frac{\partial r}{\partial u^1} \quad \text{and} \quad r_2 \equiv \frac{\partial r}{\partial u^2},
\]

which are tangent vectors at a point \(r(u^1, u^2)\) on the surface, constitute the bases of the tangent plane on the surface. The surface normal \(n\) at a point \(r(u^1, u^2)\) is given by

\[
n = r_1 \times r_2 \parallel r_1 \times r_2 \parallel,
\]

where the symbol \(\times\) denotes a cross product. In particular case where the surface is described as follows:

\[
x = u^1, \quad y = u^2 \quad \text{and} \quad z = z(u^1, u^2),
\]

we have the following relations:

\[
r_1 = (1, 0, z_1), \quad r_2 = (0, 1, z_2),
\]

\[
r_{11} \equiv \frac{\partial^2 r}{\partial u^1 \partial u^2} = (0, 0, z_{12}), \quad r_{12} \equiv \frac{\partial^2 r}{\partial u^1 \partial u^2} = (0, 0, z_{12}), \quad r_{22} \equiv \frac{\partial^2 r}{\partial u^2 \partial u^2} = (0, 0, z_{22}),
\]

and \(r_1 \times r_2 = (-p, -q, 1)\).

Here

\[
z_1 \equiv \frac{\partial z}{\partial u^1} = p, \quad z_2 \equiv \frac{\partial z}{\partial u^2} = q,
\]

\[
z_{11} \equiv \frac{\partial^2 z}{\partial u^1 \partial u^2}, \quad z_{12} \equiv \frac{\partial^2 z}{\partial u^1 \partial u^2} \quad \text{and} \quad z_{22} \equiv \frac{\partial^2 z}{\partial u^2 \partial u^2}.
\]

(Proposition 1) The constraints imposed on the gradient \(\partial E/\partial u^i\), \(i = 1, 2\), of image brightness are given by

\[
\frac{\partial E}{\partial u^i} + \sum_{j=1}^{2} \alpha^j H_{ij} = 0, \quad i = 1, 2.
\]

where \(H_{ij}, i,j=1,2\), denote the second fundamental tensor in differential geometry defined as

\[
H_{ij} \equiv -\frac{\partial n}{\partial u^i} \cdot r_j = \frac{\partial^2 r}{\partial u^i \partial u^j} \cdot n,
\]

and \(\alpha^j, j=1,2\), are given by

\[
\alpha^1 = \frac{s_1 \times r_2}{r_1 \times r_2} \quad \text{and} \quad \alpha^2 = \frac{s_1 \times r_1}{r_1 \times r_1}.
\]

Here \(s_1\) is defined as

\[
s_1 = s - (s \cdot n)n.
\]

(Proof) From eq.(9), we have

\[
\frac{\partial E}{\partial u^i} = \frac{\partial (n \cdot s)}{\partial u^i} = \frac{\partial n}{\partial u^i} \cdot s.
\]

The vector \(s_1\) in eq.(21) lies in the tangent plane and is described in terms of \(r_1\) and \(r_2\) as follows:

\[
s_1 = \alpha^1 r_1 + \alpha^2 r_2.
\]

Since vectors \((s_1 - \alpha^1 r_1)\) and \(r_2\) are parallel to each other, we have

\[
(s_1 - \alpha^1 r_1) \times r_2 = 0.
\]

Hence, we obtain \(\alpha^1\) in the following:

\[
\alpha^1 = \frac{s_1 \times r_2}{r_1 \times r_2}.
\]

In the similar manner, we obtain \(\alpha^2\) in eq.(20). Since \(n\) is a unit vector, we have the following:

\[
n \cdot n = 1 \quad \text{and} \quad \frac{\partial n}{\partial u^i} \cdot n = 0.
\]

From (21), (22), (23) and (25), we obtain the following relation

\[
\frac{\partial E}{\partial u^i} = \frac{\partial n}{\partial u^i} \cdot (\alpha^1 r_1 + \alpha^2 r_2).
\]

From (19) and (26), eq.(18) is obtained.

\[
\square
\]

In the case where the surface is described as eq.(12), \(H_{ij}, i,j=1,2\), are given by

\[
H_{ij} = \frac{z_{ij}}{\sqrt{1 + p^2 + q^2}}, \quad i, j = 1, 2.
\]

2.2 Algorithm for 3-D shape inference

In the sequel, we assume that the surface is described as eq.(12). We define the constraint \(h_1(f, g)\) as

\[
h_1(f, g) \equiv E - R(f, g) = 0,
\]

which is imposed on the image brightness. We define \(h_2^j(f, g), i=1,2\), as the left-hand side of eq.(18):

\[
h_2^j(f, g) \equiv \frac{\partial E}{\partial u^i} + \sum_{j=1}^{2} \alpha^j H_{ij} = 0, \quad i = 1, 2.
\]
which is imposed on the gradient of image brightness, where \( u^1 = x \) and \( u^2 = y \). We define vectors \( m \) and \( h_k(m), k = 1, 2, \) as
\[
m \equiv (f, g)^t, \quad h_1(m) \equiv h_1(f, g)
\]
and \( h_2(m) \equiv (h_2^1(f, g), h_2^2(f, g))^t, \)
where the symbol "\(^t\)" denotes transpose operator. Eqs.(28) and (29) are rewritten by
\[
h_k(m) = 0, \quad k = 1, 2.
\]

In our method for 3-D shape inference, we estimate \( m \) by the Marquardt method using both types of constraints \( h_k(m) = 0, k = 1, 2, \) shown in the following iterative algorithm. In the sequel, the vector \( m \) is described as vector \( m = (m_i; i = 1, 2) \).

**Marquardt method with constraints \( h_k(m) = 0, k = 1, 2 \)**

At each considered point in the image the estimate \( m^{(v)} \) at the \( v \)-th iteration is improved with \( \Delta m \) in the following steps.

1. Solve the following equation with unknown vector \( \Delta m = (\Delta m_i; i = 1, 2) \):
   \[
   [w_1 J_1(m^{(v)}) + w_2 J_2(m^{(v)}) + \gamma I] \Delta m = -\partial W(m^{(v)}) / \partial m.
   \]
   where I denotes an \( 2 \times 2 \) dimensional identity matrix, and \( W(m), J_k \) and \( G_k, k = 1, 2 \), are defined as
   \[
   W(m) = \frac{1}{2} \sum_{k=1}^2 w_k J_k Q_k h_k,
   \]

   \[
   h_k = \text{positive definite symmetric matrix}
   \]

   \[
   Q_k = \text{positive definite symmetric matrix}
   \]

   \[
   w_k = \text{weight}
   \]

   \[
   J_k \equiv \frac{\partial h_k}{\partial m}, \quad \text{Jacobian},
   \]

   \[
   \partial W(m) = \sum_{k=1}^2 w_k J_k Q_k h_k
   \]

2. Improve the estimate \( m^{(v)} \) as follows
   \[
   m^{(v+1)} = m^{(v)} + \Delta m.
   \]

In the above mentioned algorithm, the initial value of \( m \) at a point on the region except the occluding boundary, is set as \( m = 0 \). The value of \( m \) at a point on the occluding boundary is known[3].

After obtaining \( (f, g) \) by the above algorithm, we can have \( (p, q) \) by eq.(4) and from eq.(3) we can obtain the height \( z \) by integrating \( p \) and \( q \). Thus, we can reconstruct 3-D shape.

3 Theoretical Consideration on Convergence

In this section, we prove that the rate of convergence in the Marquardt method becomes faster by using the new type of constraint (29) in addition to the traditional constraint (28). We define \( B(m) \) and \( F(m) \) as
\[
B(m) \equiv w_1 G_1(m) + w_2 G_2(m) + \gamma I \quad (37)
\]
and \( F(m) \equiv -\partial W(m) / \partial m \quad (38) \)

For true solution \( m^* \), \( F(m^*) = 0 \) holds. The above mentioned algorithm (31)-(36) is rewritten by
\[
m^{(v+1)} = Y m^{(v)},
\]
where
\[
Y m = m + B(m)^{-1} F(m)
\]
and \( B(m)^{-1} = [w_1 G_1(m) + w_2 G_2(m) + \gamma I]^{-1} \quad (40) \)

(Definition 1) If \( A \) and \( B \) are \( n \times n \) dimensional Hermitian matrices such that \( A - B \) is positive definite.

(Definition 2) If \( \gamma \) is the Frechet- (or F-) differentiable at \( x \in D \) if there is an \( F^*(x) \in L(R^m, \mathbb{R}^m) \) such that
\[
\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left\| F(x + \lambda h) - F(x) - F^*(x) h \right\| = 0.
\]

The linear operator \( F^*(x) \) is called the F-derivative of \( F \) at \( x \).

\[
Y \equiv S - m R^m, \quad Y m = m + B(m)^{-1} F(m), \quad \forall m \in S,
\]
is well defined; moreover, \( Y \) is F-differentiable at \( m^* \), and its F-derivative is given by
\[
Y^*(m^*) = I + B(m^*)^{-1} F^*(m^*)
\]
Here, \( S(m^*, \delta) \) denotes the closure of an open neighborhood
\[
S(m^*, \delta) \equiv \{ m \in R^m \mid \| m - m^* \| < \delta \}.
\]

(Proof) We assume that the mapping \( B: S_0 \rightarrow L(R^m, \mathbb{R}^m) \) defined in the neighborhood \( S_0 \) of the true solution \( m^* \) of \( F(m) = 0 \), and is continuous at \( m^* \) for which \( B(m^*) \) is invertible. We set \( \beta \)
\[ B(m) - B(m') \leq \varepsilon, \quad m \in S = S(m', \delta). \] (44)

\[ B(m) \text{ is invertible at } m \in S, \text{ and satisfies the relation}[5]: \]
\[ \| B(m)^{-1} \| < \beta/(1 - \beta \varepsilon), \quad m \in S. \] (45)

Hence the mapping Y is well-defined in S.

Since \( F(m) \) is F-differentiable at \( m^* \), we can choose \( \delta > 0 \) so that
\[ \| F(m) - F(m') - F'(m')(m - m') \| < \varepsilon \| m - m' \|, \quad m \in S. \] (46)

The relation \( m^* = Ym^* \) and eq.(44) through eq.(46) lead to the following relations:
\[ \| Ym - Ym^* - [I + B(m)^{-1} F'(m')](m - m') \| \leq 2\beta(1 + \| F'(m') \|) \| m - m' \|, \quad m \in S. \] (47)

Since \( \varepsilon \) is arbitrary, and \( \| F'(m') \| \) and \( \beta = \| B(m)^{-1} \| \) are fixed values, the relation (47) implies the following. The mapping Y is F-differentiable at \( m^* \), and the F-derivative of Y is given by eq.(43).

\[ \square \]

(Proposition 3) Let \( \Pi_1 \) be the iterative process (39)-(41) with \( w_1 = 1 \) and \( w_2 > 0 \). Let \( \Pi_2 \) be the iterative process (39)-(41) with \( w_1 = 1 \) and \( w_2 = 0 \). If the spectral radius, which is defined as the maximal absolute value of eigenvalue, of \( Y'(m^*) \) in case of \( w_1 = 1 \) and \( w_2 = 0 \) is less than 1, then \( \Pi_1 \) and \( \Pi_2 \) converge. Furthermore, \( \Pi_1 \) is R-faster than \( \Pi_2 \).

(Proof) Since the i-th component of eq.(38) is represented by \( (F(m))_i = -\frac{\partial W}{\partial m_i}, \ i = 1, 2, \) the matrix \( F'(m) \) is described as
\[ F'(m) = -\frac{\partial^2 W}{\partial m^2}. \] (48)

From eq.(32), we have the following relation.
\[ W(m) = \frac{1}{2} \sum_{k=1}^{2} w_k (h_k(m^*) + J_k(m - m^*) + O((m - m^*)^2)) \]
\[ \times Q_k(h_k(m^*) + J_k(m - m^*) + O((m - m^*)^2)) \approx \frac{1}{2} \sum_{k=1}^{2} w_k (m - m^*)! J_k Q_k J_k (m - m^*). \] (49)

Hence, \( \frac{\partial^2 W}{\partial m^2} \) is given by
\[ \frac{\partial^2 W}{\partial m^2} = \sum_{k=1}^{2} w_k J_k Q_k J_k = \sum_{k=1}^{2} w_k G_k. \] (50)

Since \( Q_k, k=1,2, \) are positive definite, we have
\[ G_k = J_k Q_k J_k \geq 0, \ k = 1, 2. \] (51)

Eqs.(48),(50) and (51) lead to the following relations.
\[ \frac{\partial^3 W}{\partial m^3} \geq 0 \quad \text{and} \quad F'(m) \leq 0. \] (52)

We define \( \tilde{Y}, \tilde{B}, \tilde{F}, \tilde{F}', \tilde{Y}' \) and \( \tilde{W} \) as Y, B, F, F', Y' and W for \( w_1 = 1, w_2 = 0 \) in eq.(40), respectively. Then \( \tilde{Y}'(m^*) - Y'(m^*) \) is represented by
\[ \tilde{Y}'(m^*) - Y'(m^*) = \gamma [G_1(m^*) + \gamma I]^{-1} - \gamma \sum_{k=1}^{2} w_k G_k(m^*) + \gamma I]^{-1}. \] (53)

Define \( \tilde{A} \) and \( \tilde{B} \) as \( \tilde{A} = [G_1(m^*) + \gamma I]^{-1} \) and \( \tilde{B} = [\sum_{k=1}^{2} w_k G_k(m^*) + \gamma I]^{-1} \), respectively. Then, we have
\[ \tilde{B} \tilde{A}^{-1} = I + [G_1(m^*) + \gamma I]^{-1}[(w_1 - 1) G_1(m^*) + w_2 G_2(m^*)]. \] (54)

Denote by \( \mu \) and u, the eigenvalue and eigenvector of \( \tilde{B} \tilde{A}^{-1} \), respectively. Then, we have the following:
\[ [(w_1 - \mu) G_1(m^*) + w_2 G_2(m^*) + (1 - \mu) \gamma I] u = 0. \] (55)

Hence, \( \mu \) should satisfy
\[ \det[(w_1 - \mu) G_1(m^*) + w_2 G_2(m^*) + (1 - \mu) \gamma I] = 0. \] (56)

On the other hand, since \( G_k = J_k^2 Q_k J_k \geq 0, k=1,2, \) from eq.(51), if \( \mu < w_1 \) and \( \mu < 1 \), then
\[ \det[(w_1 - \mu) G_1(m^*) + w_2 G_2(m^*) + (1 - \mu) \gamma I] > 0. \] for \( w_2 > 0 \).

This contradicts eq.(56). Therefore, for \( w_1 = 1, w_2 = 0, \mu \) should satisfy \( \mu \geq 1 \). This implies the following where \( \lambda_i(A) \) denotes the i-th greatest eigenvalue of A.
\[ \max \lambda_i(\tilde{B} \tilde{A}^{-1}) \leq 1 \quad \text{for} \quad w_1 = 1, w_2 = 0. \] (57)

Eq.(57) leads to the following[6]:
\[ [G_1(m^*) + \gamma I]^{-1} - [\sum_{k=1}^{2} w_k G_k(m^*) + \gamma I]^{-1}, \quad \text{for} \quad w_1 = 1, w_2 = 0. \] (58)

Eqs.(53) and (58) yield
\[ Y'(m^*) \geq \tilde{Y}'(m^*), \quad \text{for} \quad w_1 = 1, w_2 > 0. \] (59)

Hence, we have[6]
\[ \lambda_i[Y'(m^*)] \geq \lambda_i[\tilde{Y}'(m^*)], \quad i = 1, 2, \quad \text{for} \quad w_1 = 1, w_2 > 0. \] (60)

Since \( \gamma > 0 \), from eqs.(37) and (50) we obtain[6]
\[ \lambda_i[Y'(m^*)] > 0, \quad i = 1, 2. \] (61)

Denote by \( \nu \) and u the eigenvalue and eigenvector of \( \tilde{B}(m^*)^{-1} \frac{\partial^3 W}{\partial m^3} \) for the particular case of \( w_1 = 1 \) and \( w_2 = 0 \), respectively. Then the following holds:
\[ \det[G_1(m^*) - \frac{\nu \gamma}{1 - \nu}] = 0. \] (62)
From eq. (51), \( G_i(m^*) \geq 0 \) and all the eigenvalues of \( G_i(m^*) \) are nonnegative. Then, we have
\[
\frac{\nu^2}{1 - \nu^2} \geq 0. 
\] (63)

Since \(|\nu| < 1\), eq. (63) and the assumption of \( \gamma > 0 \) yield
\[
0 \leq \nu < 1. 
\] (64)

On the other hand, the eigenvalue \( \hat{\mu} \) of \( \hat{Y}^i(m^*) \) satisfies
\[
\text{det}(\hat{B}(m^*)^{-1} \frac{\partial^2 \hat{Y}^i(m^*)}{\partial m_i^2} - (1 - \hat{\mu})I) = 0. 
\] (65)

Since the eigenvalue \( \nu \) of \( \hat{B}(m^*)^{-1} \frac{\partial^2 \hat{Y}^i(m^*)}{\partial m_i^2} \) satisfies eq. (64), the relation (65) implies
\[
0 \leq 1 - \hat{\mu} < 1, \text{ i.e. } 0 < \hat{\mu} \leq 1. 
\] (66)

Hence, we have
\[
0 < \lambda_i[Y^i(m^*)] \leq 1, \text{ for } i = 1, 2. 
\] (67)

Eqs. (60), (61) and (67) lead to
\[
0 < \lambda_i[Y^i(m^*)] \leq \lambda_i[Y^i(m^*)] \leq 1, \text{ for } w_1 = 1, w_2 > 0. 
\] (68)

Therefore, we have the following relation of the spectral radius \( \rho(Y^i(m^*)) \) of \( Y^i(m^*) \):
\[
0 < \rho(Y^i(m^*)) \leq \rho(Y^i(m^*)) \leq 1, \text{ for } w_1 = 1, w_2 > 0. 
\] (69)

If \( \rho(Y^i(m^*)) \) < 1, then, the iterative process (39)-(41) converges and the R-factor[5] in case of \( w_1 = 1 \) and \( w_2 > 0 \) is not greater than that in case of \( w_1 = 1 \) and \( w_2 = 0 \). Hence, \( \Pi_1 \) is R-faster[5] than \( \Pi_2 \).

![Fig. 2 Image brightness](image)

Fig. 2 Image brightness

Fig. 3 Reconstructed 3-D shape.

Figure 1 shows the convergences in \( f \) and \( g \) in the case of ellipsoidal object with radii of 30, 25 and 25. Here, \( f_s \) and \( g_e \) are set as \( f_s = 0.15 \) and \( g_e = 0.0 \). It is found that the rate of convergence in case of \( w_2 > 0 \) is faster than that in case of \( w_2 = 0 \). The new constraint can be seen as a driving term for the objective function that is of higher-order than the standard constraint based directly on the image irradiance equation. Therefore, the rate of convergence is improved. Figures 2 and 3 show the image brightness and the 3-D shape reconstructed by the proposed algorithm in section 2.2, respectively.

4 Conclusions

We have derived a new type of constraint imposed on surface normal and gradient of image brightness, and proposed 3-D shape inference algorithm by using both the traditional type of and the new type of constraints. We have proved that the R-factor is reduced by using the new type of constraint in addition to the traditional constraint, that is, using the new type of constraint is effective to make the rate of convergence of the algorithm R-faster.

5 References