A New Approach to AC Circuits Numerical Simulation

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ABSTRACT

This paper proposes a new approach to ac circuits numerical simulation. After introducing spiral vector theory which is a new ac theory, I apply this new theory to ac circuits numerical simulation and propose a new approach that uses spiral vector state variables in both steady and transient state situation. Furthermore, with an ac RLC series circuit example, I make comparisons between the general solution of spiral vector theory and the numerical solution of the new approach. The result shows that the new approach is effective in ac circuits numerical simulation.

1. INTRODUCTION

The conventional ac theory is treated as two parts. One part whose state variables are phasors is applied to steady analysis. Another part whose state variables are instantaneous real values is applied to transient analysis. It is awkward to combine two parts' results. Furthermore, I point out that in the conventional ac theory, there is no instantaneous reactive power definition which is becoming more and more important today.

These problems can be solved by spiral vector theory which is a new ac machines and circuits theory proposed by Dr. S. Yamamura[1]. (Because I only discuss lumped constant ac circuits simulation here, spiral vector theory means spiral vector ac circuits theory and is called SV theory in the paper). SV theory uses spiral vector state variables which are rotating counter-clockwise in the complex plane. SV theory unifies ac steady and transient analysis. Because spiral vectors are time-variant complex numbers, I consider that it is possible to define instantaneous reactive power.

Though SV theory provides general solutions for ac circuits, numerical simulation is still necessary and important. This paper applied SV theory to ac circuits numerical simulation. I propose a new approach that inheriting the idea from EMTP[2] that a inductance or a capacitance be equivalent to a current source and a resistance with the trapezoidal rule of integration. The difference between the new approach and EMTP is that the former uses spiral vector state variables all the time, the latter uses phasor state variables in steady state situation and instantaneous real values state variables in transient state situation. The paper is organized as follows.

After introducing SV theory, I propose the new approach. Then with an ac RLC series circuit example, I make comparisons between the general solution of spiral vector theory and the numerical solution of the new approach. At last, I give the paper conclusions.

2. SV THEORY TO AC CIRCUITS ANALYSIS

In this section, at first, I introduce the procedure of SV theory. Then I define ac power in single phase according to SV theory.

2.1. The procedure of SV theory[1]

The procedure of SV theory is shown in Figure 1. I explain each step next.

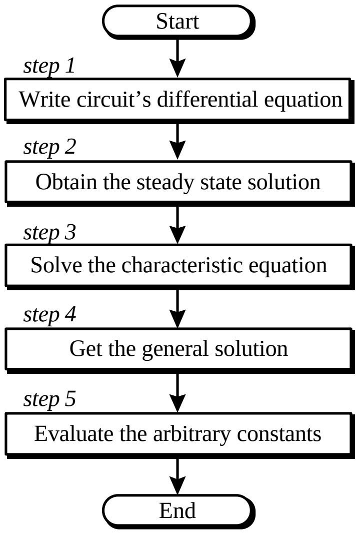

Figure 1 The procedure of SV theory

Step 1: Write circuit's differential equation

According to Kirchhoff's law, the differential equation of a two-terminal circuit can be written as

$$A(p)i = B(p)v \qquad (2\text{-}1)$$

Here, i is the current, v is the voltage, $p=d/dt$ is the differential operator. $A(p)$, $B(p)$ are polynomials as follows:

$$\left.\begin{aligned} A(p) &= a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n \\ B(p) &= b_0 + b_1 p + b_2 p^2 + \cdots + b_m p^m \end{aligned}\right\} \qquad (2\text{-}2)$$

Step 2: Obtain the steady state solution

The differential operator p becomes $j\omega$ in a steady state situation. The steady state solution of Eq.2-1 can be obtained as

$$\dot{I}_s = \frac{B(j\omega)}{A(j\omega)}\dot{V} \qquad (2\text{-}3)$$

Here, $\dot{I}_s, \dot{V}$ are circular vectors which are rotating counter-clockwise in the complex plane.

Step 3: Solve the characteristic equation

From Eq.2-1, the characteristic equation can be obtained as

$$A(p) = 0 \qquad (2\text{-}4)$$

This polynomial can be solved fast.

Step 4: Get the general solution

If all roots of the characteristic equation are real values as $d_1, d_2, \cdots, d_n$, the transient solution is

$$i_t = A_1 e^{d_1 t} + A_2 e^{d_2 t} + \cdots + A_n e^{d_n t} \qquad (2\text{-}5)$$

Here, $A_1, A_2, ..., A_n$ are arbitrary constants, which are to be determined from the initial conditions.

If some roots of the characteristic equation are complex numbers, only positive imaginary part complex numbers are treated as solutions. In these cases, initial conditions should be considered in real number set.

The general solution is the combination of steady and transient solutions as

$$i = \sqrt{2}\dot{I}_s + i_t \qquad (2\text{-}6)$$

The instantaneous real value current is the real part of the complex number solution as

$$i_{real} = \mathrm{Re}(i) \qquad (2\text{-}7)$$

Step 5: Evaluate the arbitrary constants

Substituting initial conditions into the general solutions, arbitrary constants can be evaluated.

2.2.The definitions of ac power by SV theory

Here, I define ac instantaneous power in single phase according to SV theory. The instantaneous difference-frequency power s_{df}, instantaneous difference-frequency active power p_{df}, instantaneous difference-frequency reactive power q_{df} are defined as

$$\left.\begin{aligned} s_{df} &= (1/2)(vi^*) \\ p_{df} &= (1/2)\,\mathrm{Re}(vi^*) \\ q_{df} &= (1/2)\,\mathrm{Im}(vi^*) \end{aligned}\right\} \qquad (2\text{-}8)$$

The instantaneous quasi-frequency power s_{qf}, instantaneous quasi-frequency active power p_{qf}, instantaneous quasi-frequency reactive power q_{qf} are defined as

$$\left.\begin{aligned} s_{qf} &= (1/2)(vi) \\ p_{qf} &= (1/2)\,\mathrm{Re}(vi) \\ q_{qf} &= (1/2)\,\mathrm{Im}(vi) \end{aligned}\right\} \qquad (2\text{-}9)$$

The instantaneous power s, instantaneous active power p, instantaneous reactive power q are defined as

$$\left.\begin{aligned} s &= (1/2)(vi^* + vi) = s_{df} + s_{qf} \\ p &= (1/2)\,\mathrm{Re}(vi^* + vi) = p_{df} + p_{qf} \\ q &= (1/2)\,\mathrm{Im}(vi^* + vi) = q_{df} + q_{qf} \end{aligned}\right\} \qquad (2\text{-}10)$$

Here, v is the voltage, i is the current, i^* is the conjugated number of i, all of them are spiral vectors.

The instantaneous power is the combination of difference-frequency and quasi-frequency power. The instantaneous difference-frequency power is equivalent to RMS(root-mean-square) power in a steady state situation.

The differences between SV theory and the conventional ac theory about ac power definitions are as follows:

(1)The state variables are different. SV theory uses spiral vector state variables. The conventional ac theory uses instantaneous real value state variables.

(2)The conventional ac theory does state variables transformation of three phase to two phase . SV theory doesn't do any state variables transformations.

3. A NEW APPROACH TO AC CIRCUITS NUMERICAL SIMULATION

In this section, I propose a new approach to ac circuits numerical simulation.

The procedure of the new approach is shown in Figure 2. I explain each step next.

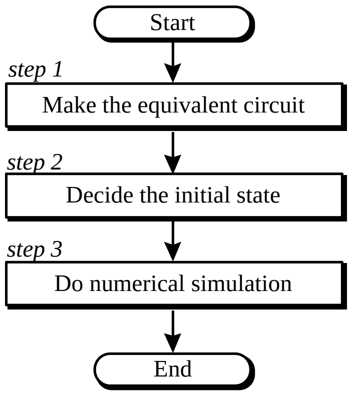

Figure 2 The procedure of the new approach

Step 1: Make the equivalent circuit

This step inherits the idea from EMTP[2] that proposed that a inductance or a capacitance can be equivalent to a current source and a resistance with the trapezoidal rule of integration.

Figure 3 shows that equivalent circuit to an inductance. Here the detailed is shown.

The voltage and current of an inductance can be written as

$$v_L(t) = L\frac{di_L(t)}{dt} \qquad (3\text{-}1)$$

Eq.3-1 can be transformed to

$$i_L(t) - i_L(t-\Delta t) = \frac{1}{L}\int_{t-\Delta t}^{t} v_L(t)dt \qquad (3\text{-}2)$$

Using the trapezoidal rule of integration, change Eq.3-2 to

$$i_L(t) = \frac{\Delta t}{2L}\{v_L(t) + v_L(t-\Delta t)\} + i_L(t-\Delta t) \qquad (3\text{-}3)$$

By distribution Eq.3-3 can be written

$$i_L(t) = \frac{1}{R_L}v_L(t) + J_L \qquad (3\text{-}4)$$

Here, the equivalent resistance and current resource are

$$\left.\begin{aligned} R_L &= 2L/\Delta t \\ J_L &= i_L(t-\Delta t) + v_L(t-\Delta t)/R_L \end{aligned}\right\} \qquad (3\text{-}5)$$

In the same way, a capacitance can be equivalent to a resistance and a current source as (Figure 4)

$$\left.\begin{aligned} R_C &= \Delta t/2C \\ J_C &= -i_C(t-\Delta t) - v_C(t-\Delta t)/R_C \end{aligned}\right\} \qquad (3\text{-}6)$$

With Eq.3-5 and Eq.3-6, present state can be calculated from history data.

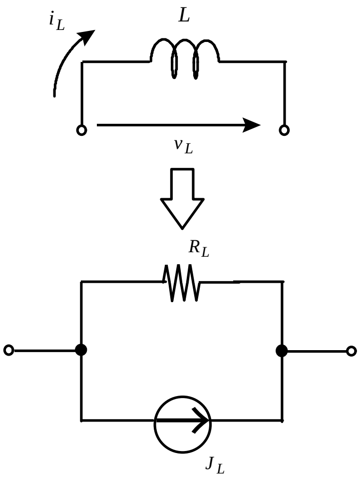

Figure 3 Equivalent circuit to an inductance

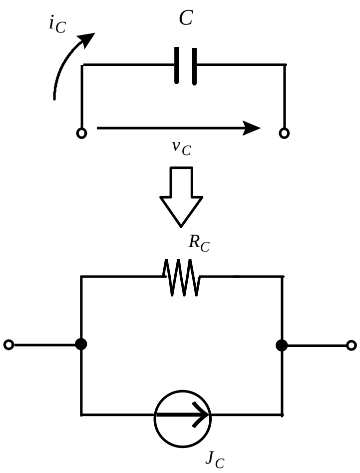

Figure 4 Equivalent circuit to a capacitance

Step 2: Decide the initial state

When all roots of the characteristic equation are real values, the state variable doesn't change instantly in complex number set, this step can be omitted.

When some roots of the characteristic equation are complex numbers, the state variable doesn't change instantly only in real number set. In this case, because the state variable changes instantly in complex number set, the initial state should be decided in the complex plane by SV theory.

Step 3: Do numerical simulation

With the equivalent circuit and the initial state of the state variable, numerical simulation can be made by nodal analysis method.

The differences between the new approach and EMTP are as follows:

(1)The state variables are different. The new approach uses spiral vector state variables. EMTP uses instantaneous real value state variables.

(2)The new approach can provide instantaneous reactive power for ac circuits. EMTP has no contents about instantaneous reactive power.

4. AN AC RLC SERIES CIRCUIT EXAMPLE

In this section, an ac RLC series circuit example is shown.

Figure 5 shows an ac RLC series circuit. Switch *S* closes at *t=0*. Find the current *i* and the ac power of the circuit. The initial conditions are that at *t=0*, flux linkage through *L* and charge across *C* are zero.

At first, make analysis by SV theory. Then do numerical simulation by the new approach. At last, numerical examples and comparisons are given.

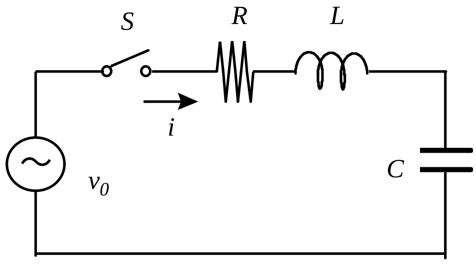

Figure 5 An ac RLC Series Circuit

4.1. Analysis by SV theory

The input voltage can be expressed as

$$v_0 = \sqrt{2}V_0 e^{j(\omega t+\varphi)} = \sqrt{2}\dot{V}_0 \qquad (4\text{-}1)$$

Here, V_0 is the effective value, φ is the initial angle of the input voltage. The instantaneous real value of the voltage is the real part of v_0 and is a cosine function. Compared to sin function which is widely used in conventional ac theory, the cosine function is convenient in defining ac power.

By writing Kirchhoff's voltage law around the circuit (for the time period after the switch *S* is closed), to obtain the differential equation

$$v_0 = Lpq^2 + Rpq + \frac{q}{C} \qquad (4\text{-}2)$$

Here, *q* is the charge of the circuit. The current of the circuit can be obtained by

$$i = pq \qquad (4\text{-}3)$$

4.1.1. Steady analysis

In the steady state situation, the differential operator *p* becomes $j\omega$, the steady state solution is obtained as

$$\dot{I} = \frac{\dot{V}_0}{R + j\omega L + \frac{1}{j\omega C}} = Ie^{j(\omega t+\theta)} \qquad (4\text{-}4)$$

The steady state solution is a circular vector. If *t=0* is substituted into Eq.4-4. the solution is equivalent to result of conventional ac theory.

4.1.2. Transient analysis

From Eq.4-2, the characteristic equation is obtained as

$$L\delta^2 + R\delta + \frac{1}{C} = 0 \qquad (4\text{-}5)$$

There are three different forms (overdamped, critically damped, underdamped) of the solution, determined by the values of *R*, *L*, and *C*. The detailed is discussed next.

A. For the overdamped ($R > 2\sqrt{\frac{L}{C}}$)

The roots of characteristic are real values as follows:

$$\delta_{1,2} = -\frac{R}{2L} \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}} \qquad (4\text{-}6)$$

The charge and current general solutions are obtained as

$$q = \sqrt{2}\dot{I} / j\omega + A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} \qquad (4\text{-}7)$$

$$i = \sqrt{2}\dot{I} + A_1\delta_1 e^{\delta_1 t} + A_2\delta_2 e^{\delta_2 t} \qquad (4\text{-}8)$$

Substituting initial conditions $q=0, i=0$ into Eq.4-7 and Eq.4-8, arbitrary constants A_1, A_2 are obtained as

$$\left.\begin{aligned} A_1 &= \frac{(\frac{\delta_2}{j\omega} - 1)}{\delta_1 - \delta_2}\delta_1\sqrt{2}\dot{I} \\ A_2 &= \frac{(\frac{\delta_1}{j\omega} - 1)}{\delta_2 - \delta_1}\delta_2\sqrt{2}\dot{I} \end{aligned}\right\} \qquad (4\text{-}9)$$

Generally arbitrary constants A_1, A_2 are complex numbers.

B. For the critically damped ($R = 2\sqrt{\frac{L}{C}}$)

The root of characteristic equation is

$$\delta = -\frac{R}{2L} \qquad (4\text{-}10)$$

The charge and current general solutions are obtained as

$$q = \sqrt{2}\dot{I} / j\omega + A_1 e^{\delta t} + A_2 t e^{\delta t} \qquad (4\text{-}11)$$

$$i = \sqrt{2}\dot{I} + A_1\delta e^{\delta t} + A_2 e^{\delta t} + A_2\delta t e^{\delta t} \qquad (4\text{-}12)$$

Substituting initial conditions $q=0, i=0$ into Eq.4-11 and Eq.4-12, arbitrary constants A_1, A_2 are obtained as

$$\left.\begin{aligned} A_1 &= -\sqrt{2}\dot{I} \\ A_2 &= \delta^2(\frac{1}{j\omega} - \frac{1}{\delta})\sqrt{2}\dot{I} \end{aligned}\right\} \qquad (4\text{-}13)$$

Generally arbitrary constants A_1, A_2 are complex numbers.

C. For the underdamped ($R < 2\sqrt{\frac{L}{C}}$)

The roots of characteristic equation are as follows:

$$\delta = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - (\frac{R}{2L})^2} = -\lambda \pm j\omega' \qquad (4\text{-}14)$$

According to SV theory, only positive imaginary part complex number($-\lambda + j\omega'$) is used. The charge and current general solutions are yielded as

$$q = \sqrt{2}\dot{I} / j\omega + A e^{(-\lambda + j\omega')t + j\varphi_A} \qquad (4\text{-}15)$$

$$i = \sqrt{2}\dot{I} + A(-\lambda + j\omega')e^{(-\lambda + j\omega')t + j\varphi_A} \qquad (4\text{-}16)$$

Substituting initial conditions $\mathrm{Re}(q)=0$, $\mathrm{Re}(i)=0$ into Eq.4-15 and Eq.4-16, arbitrary constants φ_A, A are obtained as

$$\left.\begin{aligned} \varphi_A &= \tan^{-1}(\frac{\omega\lambda\cos\theta + (\lambda^2 - \omega'^2)\sin\theta}{\omega\omega'\cos\theta}) \\ A &= -\frac{\sqrt{2}I\cos\theta}{\cos\varphi_A} \end{aligned}\right\} \qquad (4\text{-}17)$$

Arbitrary constants φ_A, A are real numbers.

Substituting the current and the voltage into Eq.2-8 to Eq.2-10, the ac power of the circuit can be obtained.

4.1.3. Verity of general solutions

For checking verity of general solutions, I use the method as following: (The underdamped case is used. Other cases are the same)

From Eq.4-15 and Eq.4-16, the derivative for q and i can be obtained as follows:

$$pq = \sqrt{2}\dot{I} + A(-\lambda + j\omega')e^{(-\lambda + j\omega')t + j\varphi_A} \qquad (4\text{-}18)$$

$$pi = p^2 q = \sqrt{2}j\omega\dot{I} + A(-\lambda + j\omega')^2 e^{(-\lambda + j\omega')t + j\varphi_A} \qquad (4\text{-}19)$$

Substituting Eq.4-15, Eq.4-18 and Eq.4-19 into right side of Eq.4-2, the following equation are obtained.

$$Lpq^2 + Rpq + \frac{q}{C} = v_{new} \qquad (4\text{-}20)$$

Comparing the v_{new} and the original input voltage v_0, if they agree with each other, it provide the evidence for verity of general solutions.

4.2. Numerical simulation by the new approach

The equivalent circuit is shown in Figure 6. According to nodal analysis method, the following equation can be obtained.

$$\begin{bmatrix} \frac{1}{R} + \frac{1}{R_L} & -\frac{1}{R_L} & -\frac{1}{R} & 0 \\ -\frac{1}{R_L} & \frac{1}{R_L} + \frac{1}{R_C} & 0 & -\frac{1}{R_C} \\ -\frac{1}{R} & 0 & \frac{1}{R} & 0 \\ 0 & -\frac{1}{R_C} & 0 & \frac{1}{R_C} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_0 \\ 0 \end{bmatrix} = \begin{bmatrix} -J_L \\ J_L - J_C \\ 0 \\ J_C \end{bmatrix} \qquad (4\text{-}21)$$

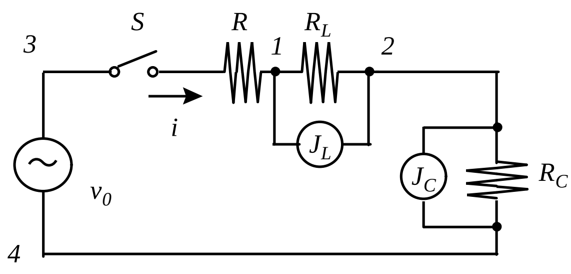

Figure 6 Equivalent Circuit for Numerical Simulation

Here, R_L , J_L is the equivalent resistance and current resource for the inductance that can be obtained by Eq.3-5. R_C , J_C is the equivalent resistance and current resource for the capacitance that can be obtained by Eq.3-6.

Transforming Eq.4-21 into the following equation.

$$\begin{bmatrix} \frac{1}{R}+\frac{1}{R_L} & -\frac{1}{R_L} \\ -\frac{1}{R_L} & \frac{1}{R_L}+\frac{1}{R_C} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -J_L \\ J_L - J_C \end{bmatrix} - \begin{bmatrix} \frac{v_0}{R} \\ 0 \end{bmatrix} \qquad (4\text{-}22)$$

Because v_1,v_2 can be obtained for Eq.4-22, the numerical simulation can be done.

4.3. Numerical Examples

In this section, I give two numerical examples. Table 1 shows the parameters of the circuit and the roots of the characteristic equation. At first, substituting these parameters into the general solutions of SV theory, then do numerical simulation by the new approach, at last compare their results each other.

Table 1 The parameters of the ac RLC series circuit

No.	*f(Hz),V₀(V),j (rad),R(W),L(mH),C(mF)*	roots
case 1	50, 100, 0, 1, 0.1, 1	-1127, -8873
case 2	50, 100, 0, 1, 1, 0.1	-500±j3122

Because the roots of the characteristic equation are real values, case 1 is an overdamped case. Figure 7 to Figure 10 show the results of the simulations in case 1.

Figure 7 shows spiral vector current in case 1. i_{ana} is the solution of SV theory. The spiral vector current starts from 0 and stabilize in a circle. According to SV theory, the state variable continues in complex plane, the initial state of the current is in the origin of the complex plane. i_{num} is the numerical solution of the new approach. i_{num} agree with i_{ana} very well.

Figure 8 shows instantaneous real value current in case 1. i_{ana}-*re* is the real part of spiral vector current i_{ana}. i_{num}-*re* is the real part of spiral vector current i_{num}. They agree with each other very well.

Figure 9 shows instantaneous difference-frequency power in case 1. $s_{df(ana)}$ is the solution of SV theory. $s_{df(num)}$ is the numerical solution of the new approach. The real axis is the instantaneous active power and the imaginary axis is the instantaneous reactive power. The instantaneous difference-frequency power starts from 0 and stabilize in a steady state point which is equivalent to RMS power. Two results agree with each other very well. Instantaneous quasi-frequency power is not shown here. The instantaneous quasi-frequency power is a oscillation element. It oscillates in double-frequency in the steady state situation,.

If initial angle of input voltage j is changed, the solutions are as follows.

(1)For the spiral vector current, though the form is the same, the direction is different in the complex plane.

(2)For the instantaneous real value current, the curve is different.

(3)For the instantaneous difference-frequency power, the curve is the same all the time.

Figure 10 shows verity of the general solutions in case 1. v_0 is the input voltage and v_{new} is calculated from the general solution of SV theory. The input voltage is a circular vector rotating counter-clockwise in the complex plane. They are agreed with each other very well. This provides evidence for verity of the general solution.

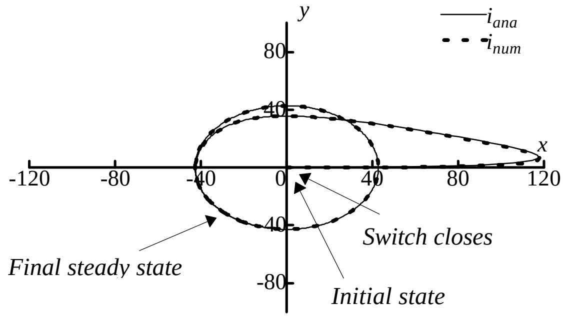

Figure 7 Spiral vector current in case 1

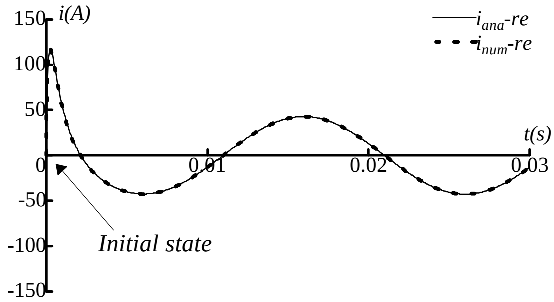

Figure 8 Instantaneous real value current in case 1

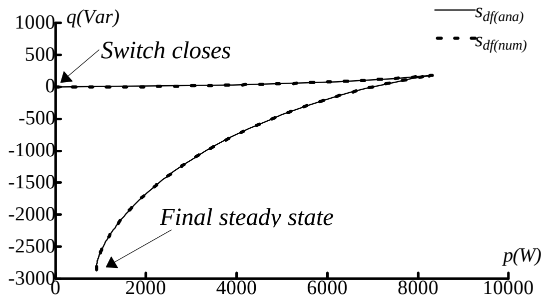

Figure 9 Instantaneous difference-frequency power in case1

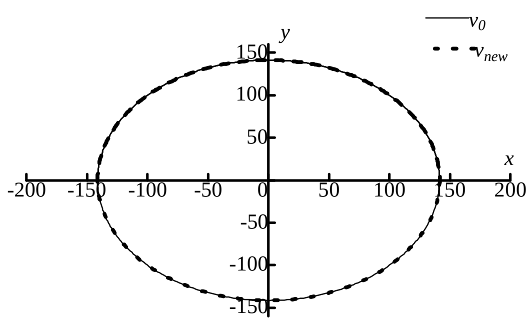

Figure 10 Verity of the general solutions in case 1

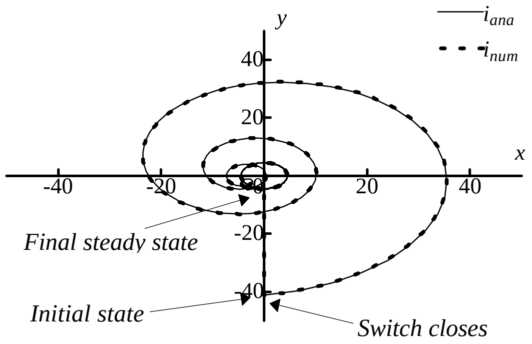

Figure 11 Spiral vector current in case 2

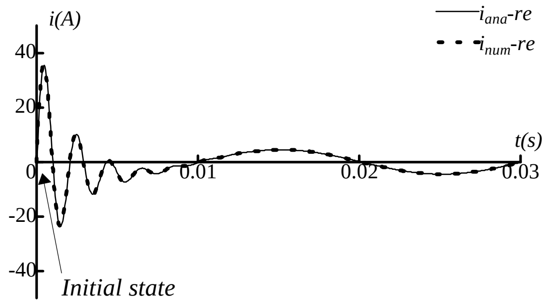

Figure 12 Instantaneous real value current in case 2

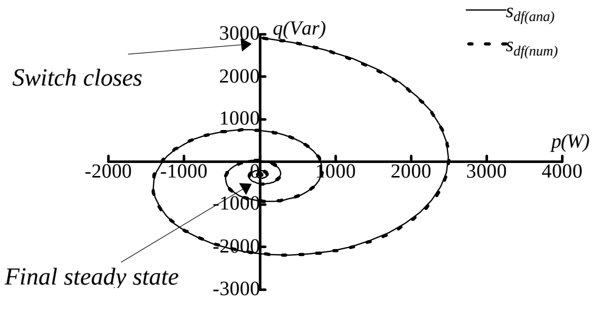

Figure 13 Instantaneous difference-frequency power in case2

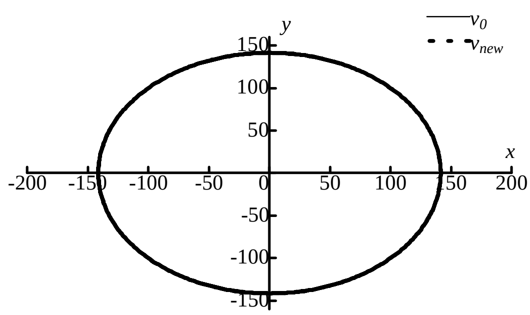

Figure 14 Verity of the general solutions in case 2

Because the roots of the characteristic equation are complex numbers, case 2 is a underdamped case. In this case, according to SV theory, only positive imaginary part of complex number (-500+j3122) is used. Figure 11 to Figure 14 show the results of the simulations in case 2.

Figure 11 shows spiral vector current in case 2. i_{ana} is the solution of SV theory. According to SV theory, the state variable changes instantly in the complex plane. With SV theory, the initial state of the current for numerical simulation is determined. i_{num} is the numerical solution of the new approach. i_{num} agree with i_{ana} very well.

Figure 12 shows instantaneous real value current in case 2. i_{ana}-*re* is the real part of spiral vector current i_{ana}. i_{num}-*re* is the real part of spiral vector current i_{num}. They agree with each other very well. Because the state variable continues in real number set, the initial state only changes in the imaginary axis of the complex plane. If the initial state is simply treated in the origin in the complex plane, thought i_{num} does not agree with i_{ana}, i_{num}-*re* does agree with i_{ana}-*re* very well.

Figure 13 shows instantaneous difference-frequency power in case 2. $s_{df(ana)}$ is the solution of SV theory. $s_{df(num)}$ is the numerical solution of the new approach. Two results agree with each other very well.

If initial angle of input voltage j is changed, the solutions are as follows.

(1)For the spiral vector current, the form and the direction is different in the complex plane.

(2)For the instantaneous real value current, the curve is different.

(3)For the instantaneous difference-frequency power, the curve is different.

Figure 14 shows verity of the general solutions in case 2. it provides evidence for verity of the general solution.

5. CONCLUSIONS

This section gives the paper conclusions.

As a new ac theory, SV theory uses spiral vector state variables in both steady and transient state situation and gives general solutions for ac circuits. According to SV theory, the instantaneous reactive power can be obtained.

Inheriting the idea from EMTP that a inductance or a capacitance be equivalent to a current source and a resistance with the trapezoidal rule of integration, and using spiral vector state variables, I propose a new approach to ac circuits numerical simulation. With an ac RLC series circuit example, I make comparisons between the general solution of spiral vector theory and the numerical solution of the new approach. The results show that the new approach can be successfully applied to ac circuits numerical simulation.

6.ACKNOWLEDGEMENT

The author would like to acknowledge Dr. S. Yamamura for his guidance in spiral vector theory.

7. REFERENCES

[1] S.Yamamura, *SV theory of AC Circuits and Machines,* Oxford: Oxford University Press, 1992.

[2] H.W.Dommel, “Digital computer solution of electromagnetic transients in single- and multi-phase networks”, *IEEE Trans. on Power Apparatus and Systems*, vol.88, no.2, pp.734-741, April 1969.