Ergodic Schrödinger operators and its related topics

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Abstract

In the first half of this lecture is to provide a basic knowledge for ergodic one dimensional Schrödinger operators which will be needed for the study of the KdV equation starting from ergodic initial data. In the course of the lecture history of this area and open problems will be also mentioned. In the second half of the lecture the construction of solutions to the KdV equation with ergodic initial data will be given.

1 Introduction

Schrödinger operators with ergodic potentials are physical models to describe the movement of electrons in matters containing impurities, which was proposed by physicist P.W. Anderson [1] in 1958.

Ergodic potential

\[
\begin{cases}
\Omega: & \text{set, } \mathcal{F}: \sigma\text{-field on } \Omega, \text{ probability measure } \mu \text{ on } (\Omega, \mathcal{F}) \\
\mathbb{R}^d \ni x \mapsto T_x : \Omega \to \Omega & \mathcal{F}\text{-measurable transformation satisfying } T_{x+y} = T_x T_y \\
\text{ergodicity: } & \mu(T_x A \cap A) = 0 \text{ for any } x \in \mathbb{R}^d \implies \mu(A) = 0 \text{ or } 1 \\
V: & \text{bounded and real valued } \mathcal{F}\text{-measurable function on } \Omega \\
\text{ergodic potential: } & q_\omega(x) = Q(T_x \omega) \quad x \in \mathbb{R}^d
\end{cases}
\]

Schrödinger operator with ergodic potential

\[L_\omega = -\Delta + q_\omega, \quad \Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_d}\]

Discrete model

\[
\begin{cases}
\Omega: & \text{set, } \mathcal{F}: \sigma\text{-field on } \Omega, \text{ probability measure } \mu \text{ on } (\Omega, \mathcal{F}) \\
\mathbb{Z}^d \ni x \mapsto T_x : \Omega \to \Omega & \mathcal{F}\text{-measurable transformation satisfying } T_{x+y} = T_x T_y \\
\text{ergodicity: } & \mu(T_x A \cap A) = 0 \text{ for any } x \in \mathbb{Z}^d \implies \mu(A) = 0 \text{ or } 1 \\
Q: & \text{bounded and real valued } \mathcal{F}\text{-measurable function on } \Omega \\
\text{ergodic potential: } & q_\omega(x) = Q(T_x \omega), \quad x \in \mathbb{Z}^d
\end{cases}
\]

\[L_\omega = -\Delta_d + q_\omega, \quad \Delta_d \text{ is discrete Laplacian}\]

Example 1 Periodic potential which corresponds to a complete crystal

\[
\begin{cases}
\Omega = \mathbb{R}^d/\mathbb{Z}^d, & \mu \text{ Lebesgue measure on } \mathbb{R}^d/\mathbb{Z}^d \\
T_x \omega = x + \omega & \text{on } \mathbb{R}^d/\mathbb{Z}^d
\end{cases}
\]
Example 2 Quasi-periodic potential
\[
\begin{aligned}
\{ & \Omega = \mathbb{R}^D / \mathbb{Z}^D, \quad \mu \text{ Lebesgue measure on } \mathbb{R}^D / \mathbb{Z}^D \\
& T_x \omega = Ax + \omega \text{ on } \mathbb{R}^d / \mathbb{Z}^d, A: D \times d \text{ matrix which is rationally independent}
\end{aligned}
\]

Example 3 Random potential
\[
\begin{aligned}
\{ & \Omega = C (\mathbb{R}^d), \quad \mu \text{ a probability measure on } C (\mathbb{R}^d) \\
& (T_x \omega) (\cdot) = \omega (\cdot + x) \text{ on } C (\mathbb{R}^d)
\end{aligned}
\]

Example of \( \mu \): For a positive definite real valued continuous function \( k(x) \) on \( \mathbb{R}^d \)
\[
\int_{\Omega} f (\omega(x_1), \omega(x_2), \cdots, \omega(x_n)) \mu (d\omega) = (2\pi \Sigma)^{-n/2} \int_{\mathbb{R}^n} f (y_1, y_2, \cdots, y_n) \exp \left( -\frac{1}{2} (\Sigma^{-1} y, y) \right) dy_1 \cdots dy_n
\]
with
\[
\Sigma = (k(x_i - x_j))_{1 \leq i, j \leq n} \quad n \times n \text{ matrix}.
\]

This \( \mu \) is called Gaussian ergodic ensemble on \( C(\mathbb{R}^d) \).

We have similar examples also for discrete potentials.
Since a solution \( u_t(x) \) to Schrödinger equation
\[
i \partial_t u_t = L \omega u_t, \quad u_0 = f
\]
gives a probability \( |u_t(x)|^2 \, dx \) for a particle to stay near \( x \) at time \( t \). Therefore, if for a bounded set \( B \subset \mathbb{R}^d \)
\[\text{(i) } \lim_{t \to \infty} \int_B |u_t(x)|^2 \, dx \quad > \quad 0 \]
\[\implies \text{the particle remains in } B \text{ with positive probability} \]
\[\implies \text{localization } \approx \text{ point spectrum} \]
\[\text{(ii) } \lim_{t \to \infty} \int_B |u_t(x)|^2 \, dx = 0 \]
\[\implies \text{the particle escapes from } B. \]
\[\implies \text{delocalization } \approx \text{ absolutely continuous spectrum.} \]

The issues we have in mind for ergodic Schrödinger operators are
(a) How does the property of ergodic potentials have an effect on the spectrum of the Schrödinger operators ?
(b) How is the dimension of the space related to the spectrum ?

The present status of the above issues is as follows.

(A) Random potentials and dimension dependence
(1) The first rigorous result was obtained by [13] in 1977 for random potentials in 1 dimension. They showed the localization (which is called now Anderson localization) for any random potentials how small it is if the space dimension is one.
In general dimension [10] succeeded in 1983 to show the Anderson localization for random potentials with big coupling constants or near the edges of the spectrum. Their method is called multi-scale analysis.

However, physicist’s conjecture that if the space dimension is larger or equal to 3, then some delocalization region appears in the middle of the spectrum has not been verified yet. This problem is supposed to be the most interesting and difficult problem in this area.

(B) **Dependence of potentials randomness or complexity**

For periodic potentials it is known that the spectrum is always purely absolutely continuous and no localization occurs. Then, the problem is to investigate what will happen if the randomness or the complexity of the potentials increases. This problem has been studied intensively especially in one dimension.

(4) In one dimension for general ergodic potentials it was shown that the vanishing set of the Lyapunov exponent coincides with the absolutely continuous (ac) spectrum (Ishii-Pastur-Kotani). And [17] proved that on the ac spectrum the reflectionless property holds, which has made it possible to show the localization under very small randomness.

(5) It has been known that almost periodic potentials indicate various aspects of the spectrum, especially for the almost Mathieu operator a complete knowledge of the spectrum was obtained.

(C) **Related problems**

(6) Spacing of the eigenvalues for random Schrödinger operators has been investigated. The results indicate that if the potential is random enough, then we have Poissonian feature in the structure of the point spectrum in any space dimension, and if the randomness decreases, then in one dimension the structure of the point spectrum is described by different probability distributions.

(7) The transition from localization to delocalization in higher dimension is tried to show from the point of view of block random matrices.

An overview for higher dimensional case, especially for multi-scale analysis and block random matrices can be found in [8], and for one dimensional case [6], [16] and [19] are good references.


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2 Weyl functions

One dimensional Schrödinger operators have functions which are called Weyl (or Weyl-Titchmarsh) functions first introduced by Weyl in 1910 and studied intensively by Titchmarsh later. The Weyl functions correspond to the potentials in a one-to-one way and they are good quantities which characterize the spectrum. The Floquet exponents for periodic potentials are invariants of the KdV equation and play a crucial role in solving the equation and the scattering data for decaying potentials can replace the role of the Floquet exponents. The Weyl functions can be defined for almost every potentials, however, they are not invariants for the KdV equation at least explicitly. In later section we express Hirota-Sato’s tau functions by the Weyl functions, which enables us to obtain a general solution to the KdV equation.

2.1 Weyl disk and boundary classification

Let

\[ q \]

be a real valued function on \( \mathbb{R} \) which is integrable on each finite interval, and the Schrödinger operator with potential \( q \) is defined by

\[ L_q = -\partial_x^2 + q. \]

The precise definition of \( L_q \) as a self-adjoint operator on \( L^2(\mathbb{R}) \) will be stated later in this section.

The first important notion is the classification of the boundaries \( \pm \infty \), which was initiated by H. Weyl in 1910. The boundaries are classified by the dimensions of the spaces

\[ W_z = \left\{ u \in L^2(\mathbb{R}_+); -u'' + qu - zu = 0 \right\}. \tag{1} \]

Since the equation in \( W_z \) is of second order, we know \( \dim W_z \leq 2 \). However, the fact \( \dim W_z \geq 1 \), equivalently \( W_z \neq \{0\} \) is not trivial. To obtain \( W_z \neq \{0\} \) Weyl introduced a disk.

Let \( \varphi_z, \psi_z \) be linearly independent solutions to

\[ -u'' + qu - zu = 0, \quad \varphi(0) = \varphi'(0) = 1, \quad \varphi'(0) = \psi(0) = 0, \tag{2} \]

where \( z \) is a complex number which will work as a spectral parameter. For \( z \in \mathbb{C}_+ \) and \( a > 0 \) set

\[ D_a(z) = \left\{ m \in \mathbb{C}_+: \int_0^a |\varphi_z(x) + m\psi_z(x)|^2 \, dx \leq \frac{\text{Im} \, m}{\text{Im} \, z} \right\}. \]

The inequality in the definition of \( D_a(z) \) can be written in a form of

\[ A + mC + \overline{m}C + \overline{B} \overline{m} \leq 0, \]

which is equivalent to

\[ \left| m + \frac{C}{B} \right|^2 \leq \left| \frac{C}{B} \right|^2 - \frac{A}{B}. \]

A little bit of calculation using \( \varphi_z(a) \psi_z'(a) - \varphi_z'(a) \psi_z(a) = 1 \) shows

\[ \begin{cases} \frac{C}{B} = \frac{\varphi_z(a) \overline{\psi_z'(a)} - \varphi_z'(a) \overline{\psi_z(a)}}{\psi_z(a) \overline{\psi_z'(a)} - \psi_z'(a) \overline{\psi_z(a)}} \\ \frac{C}{B} = \frac{A}{B} \end{cases} \]

\[ \left| \psi_z(a) \overline{\psi_z'(a)} - \psi_z'(a) \overline{\psi_z(a)} \right|^2, \]

5
which implies that \( D_a(z) \) forms a non-degenerate disk in \( \mathbb{C}_+ \) with center and radius as follows:

\[
\begin{align*}
\text{the center} \quad & = -\frac{\varphi_z(a) \psi_z'(a) - \varphi_z'(a) \psi_z(a)}{\psi_z(a) \psi_z'(a) - \varphi_z'(a) \psi_z(a)} \\
\text{the radius} \quad & = \left| \psi_z(a) \psi_z'(a) - \varphi_z'(a) \psi_z(a) \right|^{-1}.
\end{align*}
\]

We call \( D_a(z) \) as \textbf{Weyl disk}. From the definition \( D_a(z) \subset D_b(z) \) holds if \( a < b \), hence

\[ D_\infty(z) = \bigcap_{a>0} D_a(z) \]

turns to be a closed disk or one point, which is equivalently stated as follows: Let \( W_z \) be a subspace of \( L^2(\mathbb{R}_+) \) defined in (1).

- **(LD)** \( D_\infty(z) \) is a disk if and only if \( \dim W_z = 2 \).
- **(LP)** \( D_\infty(z) \) degenerates to one point if and only if \( \dim W_z = 1 \).

Therefore, automatically \( W_z \neq \{0\} \) has been proved. To know that the properties (LD) and (LP) occur simultaneously with respect to \( z \in \mathbb{C}_+ \) we define a Volterra integral operator \( V_z \) by

\[ V_z f(x) = \int_0^x (\varphi_z(x) \psi_z(y) - \varphi_z(y) \psi_z(x)) f(y) dy. \]

Then, we immediately obtain

\[ u = V_z f \iff -u'' + qu - zu = f, \quad u(0) = u'(0) = 0, \]

and

\[
\begin{align*}
\varphi_{z_2} & = (I - (z_2 - z_1) V_{z_1})^{-1} \varphi_{z_1} \\
\psi_{z_2} & = (I - (z_2 - z_1) V_{z_1})^{-1} \psi_{z_1},
\end{align*}
\]

for any \( z_1, z_2 \in \mathbb{C} \). Therefore, if \( \varphi_{z_1}, \psi_{z_1} \in L^2(\mathbb{R}_+) \) are valid, then

\[ (I - (z_2 - z_1) V_{z_1})^{-1} \]

turns out to be a bounded operator in \( L^2(\mathbb{R}_+) \), hence \( \varphi_{z_2}, \psi_{z_2} \in L^2(\mathbb{R}_+) \) hold, which implies that the properties (LD) and (LP) are equivalent to the following statements:

- **(LD)** \( \dim W_z = 2 \) holds for any \( z \in \mathbb{C} \).
- **(LP)** \( \dim W_z = 1 \) holds for any \( z \in \mathbb{C} \backslash \mathbb{R} \).

The boundary \( +\infty \) is called as of \textbf{limit circle type} if (LD) holds and of \textbf{limit point type} if (LP) occurs. The boundary \( -\infty \) has also a similar classification. A simple sufficient condition for \( q \) under which \( L_q \) satisfies (LP) is known.

**Lemma 4** If \( q \) is bounded on \( \mathbb{R}_+ \), then the boundary \( \infty \) is of limit point type.

**Proof.** If \( q \) is bounded, \( u \in W_z \) implies \( u, u'' \in L^2(\mathbb{R}_+) \). Then, in the identities

\[ \frac{1}{2} \frac{d}{dx} |u(x)|^2 = \Re \left( u'(0)\overline{u(0)} + \int_0^x u''(y)\overline{u(y)}dy \right) + \int_0^x |u'(y)|^2 dy \]
the first term of the right hand side remains bounded as \( x \to \infty \). Hence, if \( \int_{0}^{\infty} |u'(y)|^2 \, dy = \infty \), then \( |u(x)|^2 \) should be increasing from some positive number \( a \) on, which contradicts \( u \in L^2(\mathbb{R}_+) \). Therefore, we have \( u' \in L^2(\mathbb{R}_+) \).

Let \( u_1, u_2 \in W_2 \). Then, their Wronskian should be a constant, that is

\[
u_1'(x)u_2(x) - u_1(x)u_2'(x) = c.
\]

Since \( u_1, u_1', u_2, u_2' \in L^2(\mathbb{R}_+) \) are valid, \( c \) should be 0, which implies \( \dim W_2 = 1 \).

A more general condition for (LP) is

"There exists \( c > 0 \) such that \( q(x) \geq -cx^2 \) for every sufficiently large \( x \)."  

It is known that if \( q(x) = -cx^\alpha \) with some \( c > 0 \), \( \alpha > 2 \), then the boundary \( \infty \) is of limit circle type.

Suppose the boundaries \( \pm \infty \) are of limit point type for \( L_q \), then it is known that \( L_q \) can be defined as a self-adjoint operator on \( L^2(\mathbb{R}) \) uniquely without imposing any boundary condition at \( \pm \infty \). Actually, if \( q \) is bounded, then the multiplication operator \( q \) defines a bounded self-adjoint operator on \( L^2(\mathbb{R}) \). Since the Laplacian part \(-\partial_x^2\) is unitarily equivalent to the multiplication \( |x|^2 \) on \( L^2(\mathbb{R}) \) via Fourier transform, \(-\partial_x^2\) is clearly self-adjoint, hence \( L_q \) realized as their sum turns to be self-adjoint. We consider only the case when the boundaries \( \pm \infty \) are of limit point type in what follows. In this case, for any \( x \in \mathbb{C}_+ \) there exists a unique \( m_+ (z) \in D_\infty (z) \) satisfying

\[
\int_{0}^{\infty} |\varphi_+ (x) + m_+ (z) \psi_+ (x)|^2 \, dx \leq \frac{\text{Im} \, m_+ (z)}{\text{Im} \, z}.
\]

Due to \( -\varphi_+ (a) / \psi_+ (a) \in D_\infty (a) \) (observe \( \psi_+ (a) \neq 0 \)), we know

\[
- \lim_{a \to \infty} \frac{\varphi_+ (a)}{\psi_+ (a)} = m_+ (z)
\]

for any \( z \in \mathbb{C}_+ \), which shows analyticity of \( m_+ (z) \) on \( \mathbb{C}_+ \) and \( \text{Im} \, m_+ (z) > 0 \).

Set

\[
f_+ (x, z) = \varphi_+ (x) + m_+ (z) \psi_+ (x).
\]

Then, \( f_+ (x, z) \) satisfies

\[
\begin{cases}
-f_+'' (x, z) + q(x)f_+ (x, z) = zf_+ (x, z) \\
f_+ (0, z) = 1, \quad f_+ (x, z) \in L^2 (\mathbb{R}_+)
\end{cases}
\]

The above argument shows that there exists a unique solution to the equation (6) and \( f_+ (0, z) = m_+ (z) \) holds. These \( m_\pm (z) \) are called the **Weyl functions** (or Weyl-Titchmarsh functions) and will play a crucial role in the lectures.

### 2.2 Shift operation and its properties

Since we treat ergodic potentials later, we define the shift operation \( \theta_x \) for \( q \) by

\[
(\theta_x q) (\cdot) = q (\cdot + x).
\]

We denote \( m_\pm, f_\pm \) by \( m_\pm (z, q), f_\pm (x, z, q) \) respectively if it is necessary to indicate the dependence of the quantities on \( q \) explicitly.
Lemma 5 \( m_{\pm}(z, q) \) satisfy
\[
\begin{aligned}
\begin{cases}
  f_{\pm}(x, z, q) &= \exp \left( \pm \int_0^x m_{\pm}(z, \theta y) dy \right) \\
  \pm \partial_x m_{\pm}(z, \theta x) &= q(x) - z - m_{\pm}(z, \theta x) q^2
\end{cases}
\end{aligned}
\tag{7}
\]

Proof. First note \( f_{\pm}(\cdot + y, z, q) \in L^2(\mathbb{R}_\pm) \) and
\[
-f''_{\pm}(x + y, z, q) + q(x + y) f_{\pm}(x + y, z, q) = z f_{\pm}(x + y, z, q).
\]

Then, the uniqueness of the solutions implies
\[
f_{\pm}(x, z, \theta y q) = f_{\pm}(x + y, z, q) f_{\pm}(y, z, q),
\]
which yields
\[
m_{\pm}(z, \theta y q) = \pm f'_{\pm}(0, z, \theta y q) = \pm f'_{\pm}(y, z, q) f_{\pm}(y, z, q).
\tag{8}
\]

Therefore, we have the first identity of (7). The Ricatti equations in (7) can be obtained from (8) also, since \( f_{\pm} \) satisfy the second order linear differential equations.

This argument using the shift operation was employed by [15] very effectively in the study of Schrödinger operators with almost periodic potentials.

Another representation of the shifted Weyl functions can be obtained by transfer matrices. Through \( \{ \varphi_z(x, q), \psi_z(x, q) \} \) introduced in (2) define
\[
T_q(x, z) = \begin{pmatrix}
  \psi'_z(x, q) & -\varphi'_z(x, q) \\
  -\psi'_z(x, q) & \varphi'_z(x, q)
\end{pmatrix} \in SL(2, \mathbb{C}).
\tag{9}
\]

Since
\[
\begin{aligned}
\begin{cases}
  f_+(x, z, q) &= \varphi_z(x, q) + m_+(z, q) \psi_z(x, q) \\
  f_-(x, z, q) &= \varphi_z(x, q) - m_-(z, q) \psi_z(x, q)
\end{cases}
\end{aligned}
\]

holds, from (8)
\[
m_-(z, \theta x q) = -\varphi'_z(x, q) - m_-(z, q) \psi'_z(x, q)
\]
\[
-\varphi'_z(x, q) - m_-(z, q) \psi'_z(x, q)
\]
\[
= T_q(x, z) \cdot m_-(z, q)
\tag{10}
\]
is valid, where the action of \( SL(2, \mathbb{C}) \) on \( \mathbb{C} \cup \{ \infty \} \) is defined by
\[
T \cdot z = \frac{t_{11} z + t_{12}}{t_{21} z + t_{22}} \quad \text{for } T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.
\tag{11}
\]

Lemma 6 \( T_q(x, z) \) satisfies a cocycle property
\[
T_q(x + y, z) = T_{\theta y q}(x, z) T_q(y, z),
\tag{12}
\]
and if \( x \geq 0 \)
\[
T_q(x, z) \cdot w \in \mathbb{C}_+ \quad \text{if } w \in \mathbb{C}_+.
\tag{13}
\]
Proof. Let $H_q(x, z) = \begin{pmatrix} 0 & z - q(x) \\ -1 & 0 \end{pmatrix}$. Then, we have

$$\frac{d}{dx} T_q(x, z) = \begin{pmatrix} \varphi'_{\overline{z}}(x, q) & -\varphi'_{\overline{z}}(x, q) \\ -\varphi'_{\overline{z}}(x, q) & \varphi'_{\overline{z}}(x, q) \end{pmatrix} = H_q(x, z) T_q(x, z).$$

The matrix

$$T(x) = T_q(x + y, z) T_q(y, z)^{-1}$$

satisfies

$$\frac{d}{dx} T(x) = \frac{d}{dx} T_q(x + y, z) T_q(y, z)^{-1} = H_{\theta_{xy}}(x, z) T(x)$$

with $T(0) = I$, hence the uniqueness of the equation shows

$$T(x) = T_{\theta_{xy}}(x, z),$$

which is (12).

Generally $T \in SL(2, \mathbb{C})$ maps $\mathbb{C}_+$ to $\mathbb{C}_+$ if $(JT w, T w) > 0$ holds for any $w \in \mathbb{C}_+$, where

$$J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w \\ 1 \end{pmatrix}.$$

A simple calculation

$$\frac{d}{dx} \left((JT_q(x, z) w, T_q(x, z) w)\right) = \left((JH + H^* J) T_q(x, z) w, T_q(x, z) w\right)$$

shows

$$(JT_q(x, z) w, T_q(x, z) w)$$

for $z, w \in \mathbb{C}_+$, hence we have (13).

A holomorphic function on $\mathbb{C}_+$ with positive imaginary part is called as Herglotz function, and Weyl functions $m_{\pm}$ are Herglotz functions. Any Herglotz function $m$ has a representation

$$m(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \sigma(d\lambda)$$

with $\alpha \in \mathbb{R}, \beta \geq 0$ and a measure $\sigma$ on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{1}{1 + \lambda^2} \sigma(d\lambda) < \infty.$$

We call this measure $\sigma$ as the spectral measure of the Herglotz function $m$. Since the Weyl functions $m_{\pm}$ are of Herglotz, their spectral measures $\sigma_{\pm}$ are
called **spectral measures or spectral functions** for $L_q^\pm$ which are defined as restriction of $L_q$ on $L^2(\mathbb{R}_\pm)$ with Dirichlet boundary condition at $x = 0$. Especially supp $\sigma_\pm$ coincide with the spectra of the restricted $L_q^\pm$.

The **Green functions** $g_+^+(x, y), g_2(x, y)$ for the operators $L_q^+, L_q$, which are the integral kernels of $(L_q^+ - z)^{-1}, (L_q - z)^{-1}$ are given respectively by

\[
\begin{align*}
g_+^+(x, y) &= g_2^+(y, x) = \frac{\psi_2(x)f_+(y, z)}{\text{Wrons}(\psi_2, f_+)} = \psi_2(x)f_+(y, z) \\
g_2(x, y) &= g_2(y, x) = \frac{f_+(x, z)f_-(y, z)}{\text{Wrons}(f_+, f_-)} = \frac{f_+(x, z)f_-(y, z)}{m_+(z) + m_-(z)}
\end{align*}
\]

for $0 < x \leq y$ and $x \geq y$ respectively.

### 2.3 Expansion of Weyl functions when $z \to \infty$

In the discussion of the KdV equation we use an asymptotic expansion of $m_\pm(z)$ at $z = \infty$. Although the expansion we need in future is more delicate than the already known one, we present here the usual expansion, which was studied by many authors.

Atkinson observed the following fact in 1981. For the simplicity of notation we use $k = \sqrt{-z}$. If $z \in \mathbb{C}_+$, then Re $k > 0$ and Im $k < 0$. Since the radius of $D_a(z)$ has been computed in (3), $-\varphi(z)/\psi(z), m_\pm(z) \in D_a(z)$ imply

\[
\left| m_+(z) + \frac{\varphi(z)}{\psi(z)} \right| \leq \frac{2}{\psi(z) \psi'(z) - \psi'(z) \psi(z)}.
\]

Therefore, to obtain an asymptotic behavior of $m_\pm(z)$ as $z \to \infty$, we have only to know the asymptotic behavior of $\varphi(z), \psi(z)$ for a fixed $a > 0$. To this end we consider the following integral equation

\[
U(x, k) = e^{kx} + \int_0^x \frac{\sinh k(x - y)}{k} q(y) U(y, k) dy,
\]

which determines the $\varphi, \psi$:

\[
\varphi(z) = \frac{U(x, k) + U(x, -k)}{2}, \quad \psi(z) = \frac{U(x, k) - U(x, -k)}{2k}.
\]

$U(x, k)$ can be solved by an iteration:

\[
\begin{align*}
U_n(x) &= \int_0^x \frac{\sinh k(x - y)}{k} q(y) U_{n-1}(y) dy \quad \text{for } n \geq 1, \quad U_0(x) = e^{kx} \\
U(x, k) &= \sum_{n=0}^\infty U_n(x)
\end{align*}
\]

Since $|1 - e^{-2k(x-y)}| \leq 2$ if $x - y > 0$ and Re $k > 0$, we have

\[
|e^{-kx} U_n(x)| \leq |2k|^{-1} \int_0^x |1 - e^{-2k(x-y)}||q(y)||e^{-ky} U_{n-1}(y)| dy
\]

\[
\leq |k|^{-1} \int_0^x |q(y)||e^{-ky} U_{n-1}(y)| dy.
\]
Substituting this expansion to the Ricatti equation (7) yields

\[ e^{-kx}U_n(x) \leq \frac{1}{n!} \left( |k|^{-\frac{1}{2}} Q(x) \right)^n \]

\[ e^{-kx}U_n'(x) \leq \int_0^x \frac{1 + e^{-2k(x-y)}}{2} \left| q(y) \right| e^{-kqU_n-1(y)} dy \leq \frac{|k|}{n!} \left( |k|^{-\frac{1}{2}} Q(x) \right)^n \]

with \( Q(x) = \int_0^x |q(y)| dy \), hence

\[
\begin{cases}
|U(x,k) - e^{kx}| \leq \sum_{n=1}^\infty |U_n(x)| & \leq |e^{kx}| \left( \exp \left( |k|^{-\frac{1}{2}} Q(x) \right) - 1 \right) \\
|U'(x,k) - ke^{kx}| \leq \sum_{n=1}^\infty |U_n'(x)| & \leq |e^{kx}| |k| \left( \exp \left( |k|^{-\frac{1}{2}} Q(x) \right) - 1 \right)
\end{cases}
\]

Therefore, we have

\[
\begin{align*}
|\varphi_z(x) - \cosh kx| &= O \left( k^{-1} e^{kx} \right) \\
|\varphi_z'(x) - k \sinh kx| &= O \left( e^{kx} \right) \\
|\psi_z(x) - k^{-1} \sinh kx| &= O \left( k^{-2} e^{kx} \right) \\
|\psi_z'(x) - \cosh kx| &= O \left( k^{-1} e^{kx} \right)
\end{align*}
\]

which implies

\[
\psi_z(a) \overline{\psi_z'(a)} - \psi_z'(a) \overline{\psi_z(a)} = 2i \operatorname{Im} \left( k^{-1} \sinh k a \cosh \overline{k} a \right) + O \left( k^{-2} e^{2a \Re k} \right) = 2^{-1} e^{2a \Re k} i \operatorname{Im} k^{-1} + O \left( k^{-2} e^{2a \Re k} \right).
\]

Therefore, if \( z \to \infty \) in a sector \( \{ \epsilon < \arg z < \pi - \epsilon \} \), which is equivalent to \( k \to \infty \) in the sector \( \left\{ -\pi/2 + \epsilon/2 < \arg k < -\epsilon/2 \right\} \) for some \( \epsilon \in (0, \pi) \), then

\[
\frac{2}{\psi_z(a) \overline{\psi_z'(a)} - \psi_z'(a) \overline{\psi_z(a)}} \sim \frac{4e^{-2a \Re k}}{\left| \operatorname{Im} k^{-1} \right|},
\]

which means the exponentially fast decay of the right hand side of (16). This observation makes it possible to reduce the asymptotic problem of \( m_+(z) \) to that of \( -\varphi_z(a)/\varphi_z'(a) \), and this is possible essentially by getting more precise estimate of (18) if we assume the smoothness of \( q \) (see [3], [9]). Their result is

\[ m_+(z) = -k \sum_{j=0}^{n+1} c_j k^{-j} + O \left( k^{-n-1} \right), \tag{19} \]

if \( q \in C^n[0, a] \) for some \( a > 0 \). To obtain a concrete expression for the coefficients \( \{c_j\}_{j \geq 0} \) we define \( \{c_j(x)\}_{j \geq 0} \) by

\[ m_+(z, \theta, q) = -k \sum_{j=0}^{n+1} c_j(x) k^{-j} + O \left( k^{-n-1} \right). \]

Substituting this expansion to the Ricatti equation (7) yields

\[-k \sum_{j=0}^{n+1} c_j(x) k^{-j} + \left( k \sum_{j=0}^{n+1} c_j(x) k^{-j} \right)^2 = q(x) + k^2 + O \left( k^{-n-1} \right),\]

11
we have
\[
\begin{align*}
c_0' &= 1, \quad c_0 = 0, \quad c_1 = 0, \quad c_1' = 2c_0c_1 = 0, \quad c_1' = c_1' + 2c_0c_2 = q \\
\frac{1}{2} c_j' - c_j + \frac{1}{2} \sum_{\ell=1}^{j-1} c_\ell c_{j-\ell} &= 0 \quad \text{for } j \geq 3
\end{align*}
\]
which is equivalent to
\[
\begin{align*}
c_0 &= 1, \quad c_1 = 0, \quad c_2 = q/2 \\
c_j &= \frac{1}{2} c_{j-1}' - \frac{1}{2} \sum_{\ell=1}^{j-1} c_\ell c_{j-\ell} \quad \text{for } j \geq 3
\end{align*}
\]
(20)

Then, \(|c_j| \geq 0\) in (19) is obtained by setting \(c_j = c_j(0)\). This formula was obtained by [27], [12]. The other Weyl function \(m_-(z)\) can be obtained similarly, since \(m_-(z)\) is nothing but \(m_+(z)\) with potential \(\tilde{q}(x) = q(-x)\). Let \(\tilde{c}_j(x) = (-1)^j c_j(-x)\). Then, \(\tilde{c}_0(x) = 1, \tilde{c}_1(x) = 0, \tilde{c}_2(x) = \tilde{q}(x)/2\) and for \(j \geq 3\)
\[
\tilde{c}_j(x) = (-1)^j c_j(-x) = \frac{1}{2} (-1)^j c_{j-1}'(-x) - (-1)^j \frac{1}{2} \sum_{\ell=1}^{j-1} c_\ell (-x) c_{j-\ell}(-x)
\]
\[
= \frac{1}{2} c_{j-1}'(x) - \frac{1}{2} \sum_{\ell=1}^{j-1} \tilde{c}_\ell \tilde{c}_{j-\ell}(x),
\]
hence \(|\tilde{c}_j(x)| \geq 0\) becomes the \(|c_j(x)| \geq 0\) for the potential \(\tilde{q}\). Therefore, we have
\[
m_-(z) = -k \sum_{j=0}^{n+1} (-1)^j c_j k^{-j} + O(k^{-n-1}).
\]

**Lemma 7** Suppose \(q \in C^{n-1}[-a, a]\) for some \(a > 0\). Then, for any \(\epsilon \in (0, \pi)\) asymptotic expansions
\[
\begin{align*}
m_+(z) &= -k - \sum_{j=1}^{n} c_j k^{-j} + O(k^{-n-1}) \\
m_-(z) &= -k - \sum_{j=1}^{n} (-1)^j c_j k^{-j} + O(k^{-n-1})
\end{align*}
\]
as \(z = -k^2 \to \infty\) in a sector \(\{ \epsilon < \arg z < \pi - \epsilon \}\) hold, where \(c_j\) is a real number \(c_j = c_j(0)\) with \(|c_j(x)| \geq 0\) determined by (20). Each \(c_j\) is a polynomial of \(|q(0), q'(0), \cdots, q^{(j-2)}(0)|\) if \(j \geq 2\).

Later we require \(q\) so that this expansion remains valid in a domain \(D\) of \(\mathbb{C}_+\) whose boundary \(\partial D\) approaches to \(\mathbb{R}_+\) at \(+\infty\).

### 2.4 Metric on potentials

For a fixed \(M > 0\) let
\[
Q^M = \{ q; q \text{ is real valued and satisfies } |q(x)| \leq M \text{ on } \mathbb{R} \},
\]
and define a metric on \(Q^M\) by
\[
d(q_1, q_2) = \sum_{n=1}^{\infty} 2^{-n} \left( \int_{-\infty}^{\infty} (q_1(x) - q_2(x)) f_n(x) dx \right)^2 \land 1
\]
where \( \{f_n\}_{n \geq 1} \) are continuous functions on \( \mathbb{R} \) with compact supports such that \( \{f_n\}_{n \geq 1} \) are dense in each \( C([-a, a]) \) the space of continuous functions on \([-a, a]\) with sup-norm. The convergence in \( Q^M \) is equivalent to the convergence of

\[
\int_{-\infty}^{\infty} q(x) f(x) dx
\]

for any continuous function \( f \) with compact support, and \( Q^M \) is compact with this metric.

**Lemma 8** \( \varphi_z(x, q), \psi_z(x, q) \) are continuous on \( C \times Q^M \), and \( m_{\pm}(z, q) \) are continuous on \( C_+ \times Q^M \).

**Proof.** Suppose \( q_n \to q \) in \( Q^M \). Generally \( \varphi_z(x, q) \) can be obtained by

\[
\varphi_z(x, q) = 1 + \int_0^x (x-y) (q(y) - z) \varphi_z(y, q) dy = \sum_{k=0}^{\infty} \phi_k(x, q),
\]

where

\[
\phi_0(x, q) = 1 \quad \text{and} \quad \phi_k(x, q) = \int_0^x (x-y) (q(y) - z) \phi_{k-1}(y, q) dy \quad \text{for} \quad k \geq 1.
\]

Then, inductively one can show the convergence \( \phi_k(x, q_n) \to \phi_k(x, q) \) uniformly on each compact interval, which leads us to the convergence \( \varphi_z(x, q_n) \to \varphi_z(x, q) \). Similarly we have \( \psi_z(x, q_n) \to \psi_z(x, q) \), which shows the first half of the statement. Since the radius \( r(a, q) \) and the center \( o(a, q) \) of the Weyl disk \( D_a(z, q) \) are described by \( \varphi_z(x, q), \psi_z(x, q) \), the Weyl disks \( D_a(z, q_n) \) of \( q_n \) converge to \( D_a(z, q) \) for each \( z \in C_+ \). For any \( \epsilon > 0 \) there exists \( a > 0 \) such that \( r(a, q) \leq \epsilon \). Then, \( D_a(z, q_n) \subset D_a(z, q) \) for this \( a \) if \( n \) is sufficiently large, and

\[
m_{\pm}(z, q_n) \in D_a(z, q) \implies |m_{\pm}(z, q_n) - m_{\pm}(z, q)| \leq 2r(a, q) \leq 2\epsilon,
\]

which shows the convergence \( m_{\pm}(z, q_n) \to m_{\pm}(z, q) \). The convergence of \( m_{\pm}(z, q_n) \to m_{\pm}(z, q) \) is similar. It is easily seen that this convergence is uniform on each compact set of \( C_+ \), which completes the proof. \( \blacksquare \)

### 3 Remling’s theorem

Remling [23, 24] revealed the relationship between the absolutely continuous spectrum for one dimensional Schrödinger operators and the reflectionless property of the Weyl functions \( m_{\pm} \). A potential \( q \) or corresponding Schrödinger operator \( L_q \) is called **reflectionless** on \( A \in B(\mathbb{R}) \) if

\[
m_{\pm}(\xi + i0, q) = -m_{\pm}(\xi + i0, q) \quad \text{for a.e.} \quad \xi \in A
\]

holds.
3.1 Estimate of transfer matrices

On \( \mathbb{C}_+ \) define a pseudo metric \( \gamma \) by

\[
\gamma (z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{\text{Im } z_1 \text{Im } z_2}}.
\]

Then it is known that

\[
\gamma \left( F(z_1), F(z_2) \right) \leq \gamma (z_1, z_2)
\]

holds for any Herglotz function \( F \). Set

\[
T_+ = \{ T \in SL(2, \mathbb{C}) ; \ T \text{ maps } \mathbb{C}_+ \text{ into } \mathbb{C}_+ \}
\]

and for \( T \in T_+ \) define a norm by

\[
\rho(T) = \sup_{z \in \mathbb{C}_+} \frac{2 \text{Im } z}{(JTz, Tz)},
\]

which is not greater than 1 due to (22). Obviously we have inequalities

\[
\left\{ \begin{array}{l}
\gamma \left( T \cdot z_1, T \cdot z_2 \right) \leq \rho(T) \gamma (z_1, z_2) \quad \text{for any } z_1, z_2 \in \mathbb{C}_+ \text{ and } T \in T_+ \\
\rho(T_2T_1) \leq \rho(T_2) \rho(T_1) \quad \text{for } T_1, T_2 \in T_+.
\end{array} \right.
\]

(23)

For later purpose we have to estimate \( \rho(T_q(x, z)) \) for the transfer matrix \( T_q(x, z) \) of (9).

**Lemma 9** Suppose \( |q(x)| \leq M \) on \( \mathbb{R} \). Then, for any compact set \( K \) in \( \mathbb{C}_+ \) there exists a positive \( \delta \) such that \( \delta < 1 \) and

\[
\rho(T_q(z, w)) \leq k^n \quad \text{for any } z \in K \text{ and } n \geq 1.
\]

**Proof.** The space of potentials \( Q^M \) is compact and \( m_{\pm}(z, q) \) is continuous on \( \mathbb{C}_+ \times Q^M \) due to Lemma 8. The continuity of \( \{ \varphi_z(y, q), \psi_z(y, q) \} \) on the compact set \( K \times Q^M \) implies that the positive quadratic functions

\[
\left\{ \begin{array}{l}
Q_1(z, w) = \int_0^1 |\varphi_z(y, q) - w \psi_z(y, q)|^2 dy \\
Q_2(z, w) = \int_0^1 |w \varphi_z(y, q) - \psi_z(y, q)|^2 dy
\end{array} \right.
\]

of \( w \) satisfy

\[
\inf_{|w| \geq 1, (z, q) \in K \times Q^M} Q_1(z, w)|w|^{-2} = \inf_{|w| \leq 1, (z, q) \in K \times Q^M} Q_2(z, w) > 0,
\]

hence there exists a constant \( c > 0 \) such that \( Q_1(z, w) \geq c |w|^2 \) holds for any \( w \in \mathbb{C} \). Then, (14) shows

\[
(JT_q(1, z) w, T_q(1, z) w) = 2 \text{Im } w + 2 \text{Im } z Q_1(z, w) \geq 2 \text{Im } w + 2c |w|^2 \text{Im } z,
\]

hence

\[
\frac{2 \text{Im } w}{(JT_q(1, z) w, T_q(1, z) w)} \leq \frac{2 \text{Im } w}{2 \text{Im } w + 2c |w|^2 \text{Im } z},
\]

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which implies
\[ \sup_{z \in K, q \in Q} \rho(T_q(1, z)) \leq \delta \]
with
\[ \delta \equiv \sup_{z \in K, w \in \mathbb{C}_+} \frac{2 \Im w}{2 \Im w + 2c|w|^2 \Im z} < 1. \]
From the cocycle property of \( T_q(x, z) \) we have
\[ T_q(n, z) = T_{\theta_{n-1}q}(1, z) T_{\theta_{n-2}q}(1, z) \cdots T_{\theta_1q}(1, z) T_q(1, z) \]
and from (12)
\[ \rho(T_q(n, z)) \leq \prod_{k=0}^{n-1} \rho(T_{\theta_kq}(1, z)) \leq \delta^n \]
is valid. ■

### 3.2 Reflectionless property on a.c. spectrum

We follow the argument made by Breimesser-Pearson and Remling. Let \( \mathcal{H} \) be the set of all Herglotz functions on \( \mathbb{C}_+ \), namely
\[ \mathcal{H} = \{ m : \mathbb{C}_+ \to \mathbb{C}_+; m \text{ is holomorphic} \}. \]
Define for \( z = x + iy \)
\[ \omega_z(S) = \Im \left( \frac{1}{\pi} \int_S \frac{1}{t-z} dt \right) = \frac{1}{\pi} \int_S \frac{y}{(t-x)^2 + y^2} dt. \]
\( \omega_z(S) \) is related with \( \gamma(z_1, z_2) \). Let \( F_t(z) = (t-z)^{-1} \in \mathcal{H} \). Then, (22) shows
\[ \gamma(F_t(z_1), F_t(z_2)) \leq \gamma(z_1, z_2), \]
hence
\[ |\omega_{z_1}(S) - \omega_{z_2}(S)| = \left| \frac{1}{\pi} \int_S \Im(F_t(z_1) - F_t(z_2)) \right| dt \]
\[ \leq \frac{1}{\pi} \int_{-\infty}^{\infty} |F_t(z_1) - F_t(z_2)| dt \]
\[ \leq \gamma(z_1, z_2) \frac{1}{\pi} \int_{-\infty}^{\infty} \sqrt{\Im F_t(z_1)} \sqrt{\Im F_t(z_2)} dt \]
\[ \leq \gamma(z_1, z_2) \left( \frac{1}{\pi} \int_{-\infty}^{\infty} \Im F_t(z_1) dt \right)^{1/2} \]
holds. Since \( \int_{-\infty}^{\infty} \Im F_t(z) dt/\pi = 1 \), we have
\[ |\omega_{z_1}(S) - \omega_{z_2}(S)| \leq \gamma(z_1, z_2) \quad \text{for} \quad z_1, z_2 \in \mathbb{C}_+. \]
Pearson [21], [22] introduced the following notion of convergence.
Definition 10 A sequence of Herglotz functions \( \{m_n\} \) converges to \( m \in \mathcal{H} \) in value distribution if
\[
\lim_{n \to \infty} \int_A \omega_{m_n(x)}(S) \, dx = \int_A \omega_{m(x)}(S) \, dx
\]
for all Borel sets \( A, S \) of \( \mathbb{R} \) such that \( |A| < \infty \).

The following two lemmas were used by Breimesser-Pearson [4], [5]. The first lemma rephrases the convergence on \( \mathbb{C}_+ \) by that on \( \mathbb{R} \).

Lemma 11 For a sequence \( \{m_n\} \) of \( \mathcal{H} \) and \( m \in \mathcal{H} \), the following statements are equivalent.
1. \( m_n \to m \) compact uniformly on \( \mathbb{C}_+ \).
2. \( m_n \to m \) in value distribution.
3. (25) holds for any bounded intervals \( A, S \).

The second one transfers the statement on \( \mathbb{R} \) to that on \( \mathbb{C}_+ \).

Lemma 12 Let \( A \in \mathcal{B}(\mathbb{R}) \) with \( |A| < \infty \). Then
\[
\lim_{y \to 0} \sup_{m \in M, S \in \mathcal{B}(\mathbb{R})} \left| \int_A \omega_{m(x+iy)}(S) \, dx - \int_A \omega_{m(x)}(S) \, dx \right| = 0.
\]

We need the following result which is a stronger statement of the limit point condition of the boundary \( +\infty \).

Lemma 13 Let \( K \) be a compact subset of \( \mathbb{C}_+ \). Then
\[
\lim_{x \to +\infty} \gamma(T_q(x,z) \cdot w_1, T_q(x,z) \cdot w_2) = 0,
\]
uniformly in \( z \in K, w_1, w_2 \in \overline{K}_+ \).

Proof. Note first the set
\[
K_1 = \{ T_q(1,z) \cdot w; \quad z \in K, w \in \overline{K}_+ \}
\]
defines a compact set of \( \mathbb{C}_+ \). Due to (12)
\[
T_q(x,z) \cdot w = T_{\theta_1 q}(x-1,z) T_q(1,z) \cdot w
\]
holds, we have only to show
\[
\lim_{x \to +\infty} \gamma(T_{\theta_1 q}(x,z) \cdot w_1, T_{\theta_1 q}(x,z) \cdot w_2) = 0,
\]
uniformly in \( z \in K, w_1, w_2 \in K_1 \). From Lemma 9 we have
\[
\rho(T_q(n,z)) \leq \delta^n \quad \text{for any } q \in \mathcal{Q}_M \text{ and } n \geq 1.
\]
Hence, for \( n \leq x \leq n + 1 \) (9) and the cocycle property imply
\[
\rho(T_q(x,z)) = \rho(T_{\theta_{x-n} q}(n,z) T_q(x-n,z)) \leq \delta^n,
\]
which shows (26). □

To state the theorem we introduce a reflection operation \( R \) on \( Q^M \) by

\[
(Rq)(x) = q(-x).
\]

Then, obviously identities

\[
\begin{aligned}
& m_+(z, Rq) = m_-(z, q), \quad \varphi_z(x, Rq) = \varphi_z(-x, q), \quad \psi_z(x, Rq) = -\psi_z(-x, q) \\
& -T_{Rq}(-x, \xi) \cdot w = T_q(x, \xi) \cdot (-\pi) \quad \text{for any } x, \xi \in \mathbb{R} \text{ and } w \in \mathbb{C}^+.
\end{aligned}
\]

hold. Let \( \Sigma^+_{ac} \) be the absolutely continuous spectrum of \( L^+_q \) (see (15)). Our theorem is

**Theorem 14** (Remling) Let \( q \in Q^M \) and \( \bar{q} = \lim_{n \to -\infty} \theta_{x_n} q \) for a sequence \( \{x_n\} \) tending to \( +\infty \). Then, \( \bar{q} \) is reflectionless on \( \Sigma^+_{ac} \), that is

\[
m_+(\xi + i0, \bar{q}) = -m_-(\xi + i0, q) \quad \text{for a.e. } \xi \in \Sigma^+_{ac}.
\]

**Proof.** All the necessary things are already prepared and all we have to do is to follow Breimesser-Pearson and Remling. [23] noted that the property (28) is equivalent to

\[
\omega_{m_+((\xi + i0, \bar{q}))}(S) = \omega_{m_-((\xi + i0, R\bar{q}))}(-S) \quad \text{for a.e. } \xi \in \Sigma^+_{ac}
\]

for any \( S \in B(\mathbb{R}) \). Fix \( \varepsilon > 0 \) and \( A \subset \Sigma^+_{ac} \) with positive Lebesgue measure. There exists a finite decomposition \( \{A_j\}_{0 \leq j \leq N} \) of \( A \) and \( m_j \in \mathbb{C}^+ \) such that

\[
A = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_N \quad \text{disjoint with } |A_0| \leq \varepsilon,
\]

and

\[
\gamma(m_-(\xi, Rq), m_j) \leq \varepsilon \quad \text{for } \forall \xi \in A_j.
\]

Here we have written \( f(\xi) = f(\xi + i0) \) for simplicity. Since \( T_{Rq}(-x_n, \xi) \in SL(2, \mathbb{R}) \) for \( \xi \in \mathbb{R} \), we have

\[
\gamma(T_{Rq}(-x_n, \xi) \cdot m_-(\xi, Rq), T_{Rq}(-x_n, \xi) \cdot m_j) = \gamma(m_-(\xi, Rq), m_j).
\]

Applying the inequality (22) we see

\[
\left| \int_{A_j} \omega_{T_{Rq}(-x_n, \xi) \cdot m_-(\xi, Rq)}(S) \, d\xi - \int_{A_j} \omega_{T_{Rq}(-x_n, \xi) \cdot m_j}(S) \, d\xi \right| \leq \varepsilon |A_j|.
\]

Now we try to deform the second integral admitting negligible errors. The identity \( \omega_z(S) = \omega_{-z}(-S) \) and (27) imply

\[
\omega_{T_{Rq}(-x_n, \xi) \cdot m_j}(S) = \omega_{T_{Rq}(x_n, \xi) \cdot m_j}(-S) = \omega_{T_q(x_n, \xi) \cdot (-\pi)}(-S).
\]

On the other hand, Lemma 12 implies that there exists \( y > 0 \) such that for any \( m \in M \)

\[
\left| \int_{A_j} \omega_{m((\xi + iy))}(-S) \, d\xi - \int_{A_j} \omega_{m((\xi))}(-S) \, d\xi \right| \leq \varepsilon |A_j|
\]
holds. Applying this estimate to \( m(z) = T_q(x_n, z) \cdot (\overline{m_i}) \) and \( T_q(x_n, z) \cdot m_-(z, \xi, q) \) yields

\[
\begin{align*}
&\int_{A_j} \omega T_q(x_n, \xi) \cdot (\overline{m_i}) (-S) \, d\xi - \int_{A_j} \omega T_q(x_n, \xi + iy) \cdot (\overline{m_i}) (-S) \, d\xi \leq \varepsilon |A_j| \\
&\int_{A_j} \omega T_q(x_n, \xi) \cdot m_-(\xi, q) (-S) \, d\xi - \int_{A_j} \omega T_q(x_n, \xi + iy) \cdot m_-(\xi + iy, q) (-S) \, d\xi \leq \varepsilon |A_j|.
\end{align*}
\]

Then, the estimate of Lemma 13

\[ \gamma(T_q(x_n, \xi + iy) \cdot m_-(\xi + iy, q), T_q(x_n, \xi + iy) \cdot (\overline{m_i})) \leq \varepsilon \]

and (24) imply

\[ \left| \int_{A_j} \omega T_q(x_n, \xi + iy) \cdot (\overline{m_i}) (-S) \, d\xi - \int_{A_j} \omega T_q(x_n, \xi + iy) \cdot m_-(\xi + iy, q) (-S) \, d\xi \right| \leq \varepsilon |A_j|. \]

Consequently, we have

\[ \left| \int_{A_j} \omega T_q(x_n, \xi + iy) \cdot (\overline{m_i}) (-S) \, d\xi - \int_{A_j} \omega T_q(x_n, \xi) \cdot m_-(\xi, q) (-S) \, d\xi \right| \leq 3 \varepsilon |A_j|. \]

Hence, Breimesser-Pearson’s result in this case is obtained from (29), (30), namely

\[ \left| \int_{A} \omega T_q(-x_n, \xi) \cdot m_-(\xi, Rq) (-S) \, d\xi - \int_{A} \omega T_q(x_n, \xi) \cdot m_-(\xi, q) (-S) \, d\xi \right| \leq 4 \varepsilon |A| + \varepsilon, \]

for every sufficiently large \( n \). Since \( m_{\pm}(z, q) \) are continuous with respect to \( q \)

\[
\begin{align*}
&\begin{cases}
T_q(-x_n, z) \cdot m_-(z, Rq) = m_-(z, \theta_{x_n} Rq) = m_+(z, \theta_{x_n} q) \to m_+(z, q) &
T_q(x_n, z) \cdot m_-(z, q) = m_-(z, \theta_{x_n} q) \to m_-(z, q)
\end{cases}
\end{align*}
\]

holds. Therefore

\[ \int_{A} \omega m_+(\xi + i0, q) (-S) \, d\xi = \int_{A} \omega m_-(\xi + i0, \overline{q}) (-S) \, d\xi = \int_{A} \omega m_-(\xi - i0, q) (-S) \, d\xi \]

is valid. Since \( A \) is an arbitrary subset of \( \Sigma_{ac}^+ \), we have \( \omega m_+(-i0, \overline{q}) (S) = \omega m_-(i0, q) (S) \) for a.e. \( \xi \in \Sigma_{ac}^+ \), which concludes (28).

**Remark 15** If we denote the absolutely continuous spectrum of \( L_-^q \) and \( L_q \) by \( \Sigma_{ac}^- \) and \( \Sigma_{ac} \) respectively, then it is known that

\[ \Sigma_{ac}^+ \cup \Sigma_{ac}^- = \Sigma_{ac} \]

holds. Suppose the spectrum \( \Sigma_{ac}^+ \) of \( L_{ac}^+ \) coincides with \([0, \infty)\) and purely absolutely continuous there. Then, for any sequence \( x_n \to \infty \) such that \( \theta_{x_n} q = \overline{q} \)
to some \( \overline{q} \in Q^M \) the above theorem says \( \overline{q} \) is reflectionless on \([0, \infty)\). Since the spectrum of \( L_{ac}^+ \) is contained in \([0, \infty)\), hence the spectrum of \( L_q \) should be in \([0, \infty)\), which together with the reflectionless property on \([0, \infty)\) implies that \( \overline{q} = 0 \) identically. Therefore, \( \theta_{x} q \to 0 \) as \( x \to \infty \), which indicates some decaying condition for the original potential \( q \). Since it is known that if a potential \( q \) satisfies

\[ \int_{0}^{\infty} |q(x)| \, dx < \infty, \]

then \( L_{ac}^+ \) has purely absolutely continuous spectrum on \([0, \infty)\). Remling’s theorem asserts that the converse statement is somewhat valid. One can say the absolutely continuous spectrum is a rare event.
4 Ergodic Schrödinger operators in one dimension

In this section we introduce two important quantities IDS and Lyapunov exponent in the theory of one dimensional ergodic Schrödinger operators. Here, instead of providing its detailed information we only present results which are related to the study of the KdV equation. The detail can be found in [6], [16], [19].

Throughout the section we assume

\[ |q_\omega(x)| \leq M \]

for any \( x \in \mathbb{R} \) and \( \omega \in \Omega \).

4.1 IDS and Lyapunov exponent

There are two quantities which describes the spectral property of ergodic Schrödinger operators \( L_{\omega} \).

4.1.1 Integrated density of states (IDS)

There are two ways of its definition. One is constructive and physical. For \( \ell > 0 \) let \( \{ \lambda_k^{\ell,\omega} \}_{k \geq 1} \) be the eigen-values of \( L_{\omega} \) restricted on \( L^2([-\ell, \ell]) \) with Dirichlet boundary condition at \( -\ell, \ell \) and set

\[
N_{\ell}^{\omega}(\lambda) = \frac{1}{2\ell} \# \left\{ k \geq 1; \; \lambda_k^{\ell,\omega} \leq \lambda \right\}.
\]

Since we are assuming the potential \( q_\omega \) is bounded by \( M \), the eigen-values \( \lambda_k^{\ell,\omega} \) satisfy \( \lambda_k^{\ell,\omega} \geq -M \), hence \( N_{\ell}^{\omega}(\lambda) \) is finite non-decreasing function on \( \mathbb{R} \) such that \( N_{\ell}^{\omega}(\lambda) = 0 \) for \( \lambda < -M \). The ergodicity of \( q_\omega \) assures the existence of the limit of \( N_{\ell}^{\omega}(\lambda) \) as \( \ell \to \infty \) for a.e. \( \omega \in \Omega \), and is denoted by \( N(\lambda) \), namely

\[
N(\lambda) = \lim_{\ell \to \infty} N_{\ell}^{\omega}(\lambda).
\]

This quantity is independent of \( \omega \) due to the ergodicity and is called as integrated density of states (IDS) for \( L_{\omega} \).

This quantity can be identified with the expectation of the spectral measure of \( L_{\omega} \) as follows. Since the Green function \( g_\omega^z(0,0) = - (m_+^z(z))^{-1} \) (see (15)) is a Herglotz function, there exists a measure \( \sigma^z \) such that

\[
g_\omega^z(0,0) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} \sigma^z(\lambda) \frac{d\lambda}{\lambda - z}.
\]

Due to the fact \( L_{\omega} \geq -M \) the Green function \( g_\omega^z(0,0) \) is analytic on \( \mathbb{C} \setminus [-M, \infty) \) and the asymptotics of Lemma 7 implies \( g_\omega^z(0,0) \) behaves like \( (2\sqrt{-z})^{-1} \) when \( z \to \infty \) suitably, hence the measure \( \sigma^z \) is supported on \( [-M, \infty) \) and satisfies

\[
\int_0^\infty \frac{1}{\lambda + 1} \sigma^z(\lambda) < \infty.
\]
Then, it is known that
\[ N(\lambda) = \mathbb{E}\sigma^\omega((-\infty, \lambda]) = \int_\Omega \sigma^\omega((-\infty, \lambda]) \mu(d\omega) \]
holds. Therefore, an identity
\[ \mathbb{E}(g^\omega_z(0,0)) = \int_{-\infty}^\infty \frac{1}{\lambda-z} dN(\lambda) \]
is valid.

The following property of \( N(\lambda) \) is known.

**Theorem 16** \( N(\lambda) \) is continuous and non-decreasing on \( \mathbb{R} \) and satisfies
\[ \Sigma^\omega \text{(the spectrum of } L_\omega \text{)} = \text{supp } dN(\lambda). \]

The IDS can be defined for higher dimensional ergodic Schrödinger operators also and the same property holds except for the continuity.

### 4.1.2 Lyapunov exponent

There is another important quantity which describes the spectral property more intimately. Let \( T_\omega(x,z) \) be the transfer matrix of (9), namely \( T_\omega(x,z) \) is the unique solution to
\[ \frac{d}{dx} T_\omega(x,z) = \begin{pmatrix} 0 & z - q_\omega(x) \\ -1 & 0 \end{pmatrix} T_\omega(x,z), \quad T_\omega(0,z) = I. \]

Then, the cocycle property (12) implies
\[ \|T_\omega(x+y,z)\| \leq \|T_{\theta_\omega}(x,z)\| \|T_\omega(y,z)\|, \]
hence the subadditive ergode theory shows the existence of the limit of \( x^{-1} \log \|T_\omega(x,z)\| \) as \( x \to \infty \), and we denote it by \( \gamma(z) \), namely
\[ \gamma(z) = \lim_{x \to \infty} \frac{1}{x} \log \|T_\omega(x,z)\|. \]

This quantity also is independent of \( \omega \) and is called as **Lyapunov exponent**. The operator norm \( \|\cdot\| \) for matrices of \( SL(2, \mathbb{C}) \) is greater or equal to 1, hence \( \|T_\omega(x,z)\| \geq 1 \) and we have \( \gamma(z) \geq 0 \). It should be noted that the limit of \( (-x)^{-1} \log \|T_\omega(x,z)\| \) for \( x < 0 \) also the same quantity. For \( A \in B(\mathbb{R}) \) we use a notation
\[ A_{\text{ess}} = \{ \lambda \in \mathbb{R}; \ |A \cap (-\epsilon + \lambda, \epsilon + \lambda)| > 0 \text{ for any } \epsilon > 0 \}, \]
where \( |\cdot| \) denotes the Lebesgue measure of \( \cdot \). Set
\[ \mathcal{Z} = \{ \lambda \in \mathbb{R}; \ \gamma(\lambda) = 0 \} \]
\[ \mathcal{Z}_s = \left\{ \lambda \in \mathcal{Z} : \ \lim_{\epsilon \to 0} \frac{N(\lambda + \epsilon) - N(\lambda - \epsilon)}{2\epsilon} = \infty \right\} \]
and let
\[ dN = dN_{\text{ac}} + dN_s \]
be the decomposition of the measure \( dN \) into the absolutely continuous part and the singular part.
Theorem 17 Let \( \Sigma_{ac}^{\omega} \) and \( \Sigma_{s}^{\omega} \) be the absolutely continuous spectrum and the singular spectrum of \( L_{\omega} \) respectively. Then
\[
\begin{aligned}
\Sigma_{ac}^{\omega} &= Z_{\text{ess}} \\
\Sigma_{s}^{\omega} &= (Z_s \cap \text{supp } dN_s) \cup ((\mathbb{R} \setminus Z) \cap \text{supp } dN)
\end{aligned}
\]
holds for a.e. \( \omega \in \Omega \). Moreover, we have the reflectionless property on \( Z \), namely
\[
m_{ac}^{\omega} (\lambda + i0) = -m_{ac}^{\omega} (\lambda + i0) \quad \text{for a.e. } \lambda \in Z
\]
holds for a.e. \( \omega \in \Omega \).

Corollary 18 The spectrum of \( L_{\omega} \) consists only of absolutely continuous part if and only if
\[
dN_s = 0 \quad \text{and} \quad dN (\mathbb{R} \setminus Z) = 0.
\]

Remark 19 Theorem 1 is closely related with Theorem 2, namely, if we regard the shift operation as a dynamical system, then Theorem 1 states the limiting behavior of the dynamical system on the absolutely continuous spectrum, and Theorem 2 mentions the property of the equilibrium state.

4.1.3 Floquet exponent

The Floquet exponent was introduced by Johnson-Moser [15] as a generalization of that of periodic differential equations. From the formula (7) we have
\[
f_{+}^{\omega} (x, z, q) = \exp \left( \int_{0}^{x} m_{+}^{\omega} (z) \, dy \right),
\]
hence the asymptotic behavior of \( f_{+}^{\omega} \) is governed by \( \mathbb{E} \left( m_{+}^{\omega} (z) \right) \) which is denoted by \( w(z) \), namely
\[
w(z) = \mathbb{E} \left( m_{+}^{\omega} (z) \right).
\]
This \( w(z) \) is called as Floquet exponent and is known to be equal to \( \mathbb{E} \left( m_{-}^{\omega} (z) \right) \) also. The importance of \( w(z) \) is in its relationship with \( N (\lambda) \) and \( \gamma (\lambda) \), namely we have

Lemma 20 (Thouless formula) For \( \lambda \in \mathbb{R} \) it holds that
\[
\begin{aligned}
\gamma (\lambda) &= -\text{Re } w(\lambda + i0) \\
N (\lambda) &= \frac{1}{\pi} \text{Im } w(\lambda + i0)
\end{aligned}
\]

The Ricatti equation (7) yields beautiful identities
\[
\begin{aligned}
w'(z) &= -\mathbb{E} \left( (m_{-}^{\omega} (z) + m_{-}^{\omega} (z))^{-1} \right) = \mathbb{E} \left( g_{z}^{\omega} (0, 0) \right) \\
-2 \text{Re } w(z) &= \mathbb{E} \left( \frac{\text{Im } z}{\text{Im } m_{+}^{\omega} (z)} \right)
\end{aligned}
\]
To measure the magnitude of reflectionlessness we define
\[
R(z) = \frac{m_{+}(z) + m_{-}(z)}{m_{+}(z) + m_{-}(z)}
\]
and call it as reflection coefficient. This quantity was first considered by Gesztesy-Simon, Rybkin and others as a generalization of the conventional reflection coefficient defined for decaying potentials. (31) implies another identity

\[- \frac{\text{Re} w(z)}{\text{Im} z} - \text{Im} w'(z) = \frac{1}{4} \mathbb{E} \left( \left( \frac{1}{\text{Im} m_+^\omega (z)} + \frac{1}{\text{Im} m_-^\omega (z)} \right)^2 \right), \tag{33}\]

which was crucial to prove Theorem 17. Set

\[\chi (z) = - \frac{\text{Re} w(z)}{\text{Im} z} - \text{Im} w'(z) > 0. \tag{34}\]

The following lemma will be used for the KdV equation.

**Lemma 21** For \( z \in \mathbb{C}_+ \) we have

\[\mathbb{E} (|R^\omega (z)|) \leq \sqrt{2 \chi (z) \text{Im} w(z)}. \]

**Proof.** Applying the Schwarz inequality to (33) yields

\[\mathbb{E} (|R^\omega (z)|) \leq \sqrt{\mathbb{E} \left( \left( \frac{1}{\text{Im} m_+^\omega (z)} + \frac{1}{\text{Im} m_-^\omega (z)} \right)^{-1} \right)} \sqrt{\chi (z)} \]

\[\leq \sqrt{\mathbb{E} (\text{Im} m_+^\omega (z) + \text{Im} m_-^\omega (z))} \sqrt{\chi (z)} = \sqrt{2 \chi (z) \text{Im} w(z)}. \]

\[\blacksquare\]

### 4.2 Moment estimate of Lyapunov exponent

For later purpose we estimate the moments of Lyapunov exponent by the derivatives of potentials \( q_\omega \), which was given by [18].

(31) implies that the Floquet exponent \( w \) satisfies \( \text{Im} w(z) > 0 \) for \( z \in \mathbb{C}_+ \), hence a product \(-w_1w_2\) of two Floquet exponents \( w_1, w_2 \) gives a Herglotz function. Taking the simple Floquet exponent \( w_0(z) = -\sqrt{-z} \) of the constant potential 0, we have a Herglotz function \( \sqrt{-zw}(z) \).

**Lemma 22** Suppose \( q_\omega (x) \in C_b^{2n} (\mathbb{R}) \) for some \( n \geq 0 \). Then

\[\int_0^\infty \lambda^n \sqrt{\chi} \gamma (\lambda) \, d\lambda < \infty \tag{35}\]

**is valid.**

**Proof.** (20) and Lemma 7 imply

\[m_+^\omega (z) + m_-^\omega (z) \]

\[= -2\sqrt{-z} - \sum_{j=1}^{2n+1} c_{j+1}^\omega \left( 1 + (-1)^{j+1} \right) \sqrt{-z}^{-j} + O \left( z^{-n-1} \right) \]

\[= -2\sqrt{-z} - 2 \sum_{1 \leq j \leq 2n+1} c_{j+1}^\omega \sqrt{-z}^{-j} + O \left( z^{-n-1} \right), \]

\[\text{for } j \text{ odd.} \]

\[\text{for } j \text{ even.} \]
where \( c^j \) is a polynomial of \( \{ q_\omega(0), q'_\omega(0), \ldots, q_{\omega(j-2)}(0) \} \), hence

\[
w(z) = -\sqrt{-z} - \sqrt{-z} \sum_{j=1}^{n+1} a_{2j} (-z)^{-j} + O(z^{-n-1})
\]  

(36)

with \( a_j = \mathbb{E}(c^j) \). On the other hand, since \( \sqrt{-z} w(z) \) is a Herglotz function and

\[
\Im \sqrt{-(\lambda + i0)} w(\lambda + i0) = \begin{cases} 
\pi \sqrt{-\lambda N}(\lambda) & \text{if } \lambda < 0 \\
\sqrt{\lambda}(\lambda) & \text{if } \lambda > 0
\end{cases}
\]

holds, we have

\[
\sqrt{-z} w(z) = \alpha + \beta z + \frac{1}{\pi} \int_{-M}^{0} \frac{\pi \sqrt{-\lambda N}(\lambda)}{\lambda - z} d\lambda \\
\quad + \frac{1}{\pi} \int_{0}^{\infty} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda^2 + 1} \right) \sqrt{\lambda}(\lambda) d\lambda
\]

with \( \alpha \in \mathbb{R} \) and \( \beta \geq 0 \). This combined with (36) implies \( \beta = 1 \) and

\[
\int_{0}^{\infty} \frac{1}{\lambda + 1} \sqrt{\lambda}(\lambda) d\lambda < \infty,
\]

hence we have

\[
w(z) = -\sqrt{-z} - \frac{a}{\sqrt{-z}} + \frac{1}{\sqrt{-z}} \int_{-M}^{0} \sqrt{-\lambda N}(\lambda) \frac{d\lambda}{\lambda - z} + \frac{1}{\pi \sqrt{-z}} \int_{0}^{\infty} \sqrt{\lambda}(\lambda) \frac{d\lambda}{\lambda - z}
\]  

(37)

Substituting this expansion into (37) yields

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{\sqrt{\lambda}(\lambda)}{\lambda - z} \frac{d\lambda}{\lambda - z} = \sum_{j=1}^{n} b_{j} (-z)^{-j} + O(z^{-n-1})
\]

as \( z \to -\infty \) with some other \( b_j \in \mathbb{R} \). Then, inductively we can show the finiteness of the \( n \)-th moment. \( \blacksquare \)

5 KdV equation

The KdV equation is an equation describing motion of waves on shallow water, and it is known that it has infinitely many invariants, which makes it possible to solve it completely for periodic or decaying initial data. However, if the initial data are neither periodic nor decaying, we cannot use the invariants effectively and only several special cases are known to be solvable. In this section we give an approach to this problem by extending the Sato’s method of Grassmann manifold, since this method admits solutions with periodic or decaying initial data in a unified way, although they are in a sense finite dimensional due to its algebraic nature.
5.1 Construction of KdV flow on reflectionless potentials

In this section we state results of [20] without proofs, which will be necessary in the later section for the construction of solutions to the KdV equation with non-periodic and non-decaying initial data.

The prototype of the tau function was introduced by Hirota [14], and later Sato [25] rewrote it by determinant and used it as a central notion in his theory. Segal-Wilson [26] developed Sato’s algebraic theory on a Grassmann manifold $Gr^{(2)}$ consisting of subspaces $W$ of $H = L^2 (|z| = \sqrt{r})$ satisfying the two conditions

\begin{align*}
(i) & \quad f \in W \implies z^2 f \in W \\
(ii) & \quad p_+ : W \rightarrow H_+ = \left\{ u \in H; \quad u(z) = \sum_{n \geq 0} u_n z^n \right\} \text{ is bijective.}
\end{align*}

Let $A_W$ be the unique operator from $H_+$ to $H \setminus H_+$ satisfying

$W = \{ f \in H; \quad f = f_+ + A_W f_+ \text{ with } f_+ \in H_+ \},$

whose existence is guaranteed by the condition (ii). Since $H_+$ is generated by multiplying $z^2$ consecutively to the two elements $\{1, z\}$, the above two properties of $W$ suggest us to introduce two elements of $H_-$

$\varphi_W(z) = (A_W 1)(z), \quad \psi_W(z) = (A_W z)(z).$

Then, without difficulty we see

$W = \langle z^{2n} (1 + \varphi_W(z)) + z^{2n} (z + \psi_W(z)) \rangle_{m,n \geq 0},$

where $\langle \cdot \rangle$ denotes the closure of all linear combinations of $\cdot$ in $H$. By these two functions define

$m_W(z) = \frac{z + \psi_W(z)}{1 + \varphi_W(z)} + a_1(W)$

where $a_1(W)$ is the first coefficients of the expansion

$\varphi_W(z) = \sum_{n=1}^{\infty} a_n(W) z^{-n}.$

$a_1(W)$ is added so that we have

$m_W(z) = z + O(z^{-1}) \quad \text{as } z \to \infty.$

Let $p_+$ be the orthogonal projection from $H$ to $H_+$, and set

$\begin{cases} 
\Gamma = \{ g; \quad g = e^h \quad h \text{ is holomorphic on } \mathbb{C} \setminus (I_s \cup iI_s) \text{ for an } s < \sqrt{r} \} \\
\Gamma_{\text{real}} = \{ g; \quad g = e^h \quad h \text{ is real valued on } \mathbb{R} \}
\end{cases}.$

For $W \in Gr^{(2)}$, $g \in \Gamma$ the tau function $\tau_W(g)$ is defined by

$\tau_W(g) = \det (I + g^{-1} p_+ g A_W),$

which is the crucial tool for the theory. Let

$\rho_W(g) = \exp \left( \frac{1}{2\pi i} \int_{|\zeta| = \sqrt{r}} h' (\zeta) \log (1 + \varphi_W(\zeta)) d\zeta \right) \quad \text{for } g = e^h$
and define
\[ \tau_{mW}(g) = \frac{\tau_W(g)}{\rho_W(g)}. \]

The tau-function satisfies the following basic properties.

**Lemma 23**

1. \( gW \in Gr^{(2)} \) if and only if \( \tau_W(g) \neq 0 \).
2. \( \tau_W(g) \) satisfies the cocycle property
   \[ \tau_W(g_1g_2) = \tau_W(g_1)\tau_{g_1W}(g_2) \quad \text{if} \quad \tau_W(g_1) \neq 0. \]
3. \( \tau_W(g) \) is decomposed as
   \[ \tau_W(g) = \rho_W(g)\tau_{mW}(g), \]
   where \( \tau_{mW}(g) \) depends only on \( mW \), namely if \( W_1, W_2 \in Gr^{(2)} \) satisfy \( mW_1 = mW_2 \), then \( \tau_{mW_1}(g) = \tau_{mW_2}(g) \).

Now we specify \( W \)'s corresponding to potentials. For a bounded potential \( q \in Q^M \) let \( m_{\pm} \) be its Weyl functions and define
\[ m = \begin{cases} -m_+(-z^2) & \text{if } \text{Re } z > 0 \\ -m_-(-z^2) & \text{if } \text{Re } z < 0 \end{cases} \]  

(38)

Then, \( m \) is holomorphic on \( \mathbb{C} \setminus \left( [-\sqrt{M}, \sqrt{M}] \cup i\mathbb{R} \right) \) and satisfies
\[ \frac{\text{Im } m(z)}{\text{Im } z} > 0 \quad \text{for } z \in \mathbb{C} \setminus \left( [-\sqrt{M}, \sqrt{M}] \cup i\mathbb{R} \right). \]

If \( m_{\pm} \) are reflectionless on \( (\lambda_1, \infty) \) for \( \lambda_1 > 0 \), then \( m \) is holomorphic on \( \mathbb{C} \setminus \left( [-\sqrt{M}, \sqrt{M}] \cup (-\sqrt{\lambda_1}, \sqrt{\lambda_1}) \right) \). Keeping this in mind, we set
\[ M_r^{refl} = \left\{ m; \begin{array}{l} (i) \ m \text{ is analytic on } \mathbb{C} \setminus (I_+ \cup iI_+) \\ (ii) \ m(z) = O(z^{-1}) \text{ as } z \to \infty \\ (iii) \ m(z) = m(\overline{z}), \ \text{Im } m(z)/\text{Im } z \geq 0 \\ (iv) \ m(x) \geq m(-x) \text{ if } x > r \end{array} \right\}, \]

where \( I_+ = [-r, r] \). (iv) is equivalent to
\[ m_+(x) + m_-(x) < 0 \quad \text{if } x > r, \]
which follows from \( \text{sp } L_q \subset [-M, \infty) \) and the identity
\[ -(m_+(x) + m_-(x))^{-1} = g_{-x}(0; 0) > 0. \]

Set
\[ W_m = \{ f; \ f(z) = \varphi(z^2) + m(z)\psi(z^2) \quad \text{with } \varphi, \psi \in H_+(|z| = r) \} . \]

Then, without difficulty we see \( W_m \in Gr^{(2)} \) and \( \varphi_{W_m} = 0, \psi_{W_m} = m - z \), hence \( mW = m \) and
\[ \tau_m(g) = \tau_{mW}(g) = \tau_{W_m}(g) / \rho_{W_m}(g) = \tau_{W_m}(g), \]

since \( \rho_{W_m}(g) = 1 \).
One can express this \( \tau_m(g) \) by a Fredholm determinant of some integral operator. In the sequel for a function \( f \) on a domain \( D \) in \( \mathbb{C} \) satisfying \( -z \in D \) if \( z \in D \) we define
\[
 f_{\pm}(z) = \frac{(f(\sqrt{z}) + f(-\sqrt{z}))}{2}, \quad f_0(z) = \frac{(f(\sqrt{z}) - f(-\sqrt{z}))}{(2\sqrt{z})},
\]
where \( \sqrt{z} \) is defined as a holomorphic function on \( \mathbb{C}\setminus(-\infty,0] \) satisfying \( \sqrt{1} = 1 \).

For \( m \in \mathcal{M}^{refl}_{\sqrt{	au}} \) we define the tau function \( \tau_m(g) \) on \( \Gamma_{\text{real}} \). Let \( C, C' \) be simple smooth closed curves surrounding the interval \([-r,r]\) which are directed anticlockwise.

Let \( \delta \) be an analytic function such that \( \delta_c, \delta_o \) are analytic on a simply connected domain containing \( C' \) and set
\[
\tilde{m}(z) = m(z) - \delta(z).
\]
Define an integral kernel \( N_{g,m}(z,\lambda) \) and its integral operator \( N_m(g) \) by
\[
\begin{cases}
 N_{g,m}(z,\lambda) = \frac{1}{2\pi i} \int_{C'} \frac{\tilde{g}_c(\lambda')(g\tilde{m})_c(\lambda') + \tilde{g}_c(\lambda')(g\tilde{m})_r(\lambda')}{(\lambda' - z)(\lambda' - \lambda)} m_o(\lambda')^{-1} d\lambda', \\
 (N_m(g)f)(z) = \frac{1}{2\pi i} \int_{C} N_{g,m}(z,\lambda) f(\lambda) d\lambda,
\end{cases}
\]
where \( \tilde{g}(z) = g(z)^{-1} \). The operator \( N_m(g) \) defines a trace class operator on \( L^2(C) \). Then, we have

**Lemma 24.** \( \tau_m(g) \) is independent of \( C, C' \) and \( \delta \), and satisfies
\( 1 \) \( \tau_m(g) = \det(I + N_m(g)) \) for \( g \in \Gamma \).
\( 2 \) \( \tau_m(g) > 0 \) holds for \( g \in \Gamma_{\text{real}} \).

The potential \( q \) can be recovered from \( m \) by
\[
 q(x) = -2\partial_x^2 \log \tau_m(e_x) \quad \text{with} \quad e_x(z) = e^{xz} \in \Gamma_{\text{real}} \quad \text{if} \quad x \in \mathbb{R}.
\]

Set
\[
 Q_{\sqrt{\tau}}^{refl} = \{ g; \ q(x) = -2\partial_x^2 \log \tau_m(e_x) \quad \text{with} \quad m \in \mathcal{M}^{refl}_{\sqrt{\tau}} \}.
\]

Then, we have a flow \( \{ K(g) \}_{g \in \Gamma_{\text{real}}} \) on \( Q_{\sqrt{\tau}}^{refl} \) satisfying
\[
\begin{cases}
 (K(e^{xz})q)(x) = q(x + t) \quad \text{a solution to} \quad \partial_t q = \partial_x q \\
 (K(e^{-4xz})q)(x) = q(x, t) \quad \text{a solution to} \quad \partial_t q = -\partial_x^2 q + 6\partial_x q \quad \text{(KdV)} \\
 (K(e^{2n+1}z)q)(x) = q_n(x, t) \quad \text{a solution to} \quad \partial_t q = \partial_x^{2n+1} q + \text{non-linear term}
\end{cases}
\]

The cocycle property (2) of Lemma 23 can be interpreted by \( \tau_m(g) \) as follows.

Set
\[
 q_{\zeta}(z) = (1 - z\zeta^{-1})^{-1} \in \Gamma \quad \text{if} \quad |\zeta| > \sqrt{\tau}.
\]
Lemma 25 Suppose $g_1, g_2 \in \Gamma$ and $\tau_m (g_1) \neq 0$. Then, we have
\[
\tau_m (g_1 g_2) = \tau_m (g_1) \tau_{m \phi_1 \phi_2} (g_2) \exp \left( \frac{1}{2\pi i} \int_{|\zeta| = r} h_2 (\zeta) \log \frac{\tau_m (g_1 \phi_2 \phi_1)}{\tau_m (g_1)} d\zeta \right) (41)
\]
with $g_2 = e^{h_2}$.

In the proof we use an identity
\[
\tau_W (q, \zeta) = 1 + \varphi_W (\zeta). (42)
\]

One can show also

Lemma 26 Suppose $m \in \mathcal{M}^{refl}_{\sqrt{\tau}}$ and $g \in \Gamma_{real}$. Then, $m \phi_2 \phi_1 \phi_2 \in \mathcal{M}^{refl}_{\sqrt{\tau}}$ holds, which derives an action $g \cdot m = m \phi_2 \phi_1 \phi_2$ on $\mathcal{M}^{refl}_{\sqrt{\tau}}$ by $\Gamma_{real}$.

The resulting space $\mathcal{Q}^{refl}_{\sqrt{\tau}}$ of potentials consists of meromorphic functions on $\mathbb{C}$ with no poles on $\mathbb{R}$, and it contains a certain kind of quasi-periodic functions as well as rapidly decaying functions, however a direct characterization of $\mathcal{Q}^{refl}_{\sqrt{\tau}}$ is not known. From the spectral theoretic point of view the associated Schrödinger operator $L_q$ has purely absolutely continuous spectrum on $(r, \infty)$ with reflectionless property and on $(-r, r)$ the spectrum can be arbitrary. Our next task is to remove the absolute continuity of the spectrum on $(r, \infty)$ by letting $r \to \infty$ in the expression of $\tau_m (g) = \det (I + N_m (g))$ of Lemma 24.

5.2 Extension of KdV flow

In this section we extend the $\tau$-function with the help of Lemma 24, and define the KdV flow on a wider class of potentials $q$.

5.2.1 Potential class

What we have in mind about the curves $C, C'$ is illustrated in the figure below.

![Diagram of curves](image.png)

The precise shapes of the curves $C, C'$ are determined depending on $g \in \Gamma$ so that $g_c, g_a$ are bounded on the curves, which will be necessary for the operator $N_m (g)$ to be a trace class, or more generally Hilbert-Schmidt class operator. Suppose the curve $C$ is parametrized as
\[
x + i (-x)^{-\alpha}
\]
for large \(-x\). For instance if \(g(z) = e^{tz^3} (t \in \mathbb{R})\), the exponent of \(g_e\) behaves as \(x \to -\infty\)

\[ e^{3/2} = \left( x + i(-x)^{-\alpha} \right)^{3/2} = -i(-x)^{3/2} - \frac{3}{2}(-x)^{-\alpha+1/2} + o\left( |x|^{-\alpha+1/2} \right), \]

hence, if \(\alpha \geq 1/2\), then \(g\) remains bounded on \(C\). More generally if \(h(z) = z^n\) with odd \(n\), we have to choose \(\alpha\) so that \(\alpha \geq n/2 - 1\). It should be remarked that \(h\) has to be an odd polynomial with real coefficients. Otherwise \(g\) might be unbounded on \(C\) whichever curve \(C\) we choose.

The boundedness of \(g_e, g_o\) is not sufficient in order that the \(N_m(g)\) may be a trace class operator, and the terms \(m_e, m_o\) should decay as \(x \to -\infty\), which is the reason why we have attached \(\delta\) to \(m\) in the definition of \(N_m(g)\).

Hereafter we fix \(\lambda_0 < 0\) and assume that the Schrödinger operator \(L_q\) has its spectrum \(\sigma(L_q) \subset [\lambda_0, \infty)\).

For a fixed \(b > \lambda_0\) let \(\omega\) be a smooth function on \((-\infty, b]\) satisfying

\[ \omega(x) > 0 \text{ for any } x \in (-\infty, b) \text{ and } \omega(b) = 0. \]

For this \(\omega\) define the curve \(C_\omega\) on \(\mathbb{C}\) by

\[ C_{\omega} \cap \mathbb{R}^+ = \{ x + i\omega(x) \mid x \leq b \}, \quad (43) \]

and on \(\mathbb{C}_-\) it is defined by taking the complex conjugate of \(C_\omega\) on \(\mathbb{C}_+\). Its direction is anti-clockwise. We assume \(\omega\) satisfies

\[ \lim_{x \to -\infty} (-x)^{n/2-1} \omega(x) \in (0, \infty) \quad (44) \]

for an \(n \geq 1\), which means that \(\omega\) depends on \(n\). The interior domain \(D^+_{\omega}\) and the exterior domain \(D^-_{\omega}\) of \(C_\omega\) are defined by

\[ \begin{cases} D^+_{\omega} = \{ z \in \mathbb{C} \mid \text{Im} z < \omega(\text{Re} z), \text{Re} z < b \} \\ D^-_{\omega} = \mathbb{C} \setminus (D^+_{\omega} \cup C_\omega) \end{cases} \quad , \quad (45) \]

which is illustrated below.

Moreover, set

\[ H(D^\pm_{\omega}) = L^2(C_\omega)\text{-closure of all rational functions with no poles in } D^\pm_{\omega}. \quad (46) \]

28
For $N > 0$ define a space $\Phi^N_n$ as

$$\Phi^N_n = \left\{ u \in H(D^-_\omega) ; \ \exists f \in H(D^+_\omega) \text{ s.t. } \int_{C^-_\omega} |z|^{2N} |u(z) - f(z)|^2 dz < \infty \right\},$$

(47)

where $n$ comes from (44). For $u \in H(D^-_\omega)$ and $f \in H(D^+_\omega)$ the Cauchy theorem implies

$$\left\{ \begin{array}{l}
  u(z) = \frac{1}{2\pi i} \int_{C^-_\omega} \frac{u(\lambda)}{z-\lambda} d\lambda \\
  0 = \frac{1}{2\pi i} \int_{C^-_\omega} \frac{f(\lambda)}{z-\lambda} d\lambda,
\end{array} \right. \text{ if } z \in D^-_\omega,$$

hence

$$u(z) = \frac{1}{2\pi i} \int_{C^-_\omega} \frac{u(\lambda) - f(\lambda)}{z-\lambda} d\lambda$$

(48)

holds. For $u \in \Phi^N_n$ set

$$\|u\|_{n,N} = \inf_{f \in H(D^+_\omega)} \sqrt{\int_{C^-_\omega} |z|^{2N} |u(z) - f(z)|^2 dz}.$$  

(49)

Suppose $\|u\|_{n,N} = 0$. Then, $u = 0$ follows from (48) easily. The other properties of norms are clear, hence $\|\|_{n,N}$ defines a norm in $\Phi^N_n$. It should be remarked that $\Phi^N_n$ depends on $\omega$, namely, even if $\omega_1, \omega_2$ satisfy

$$\lim_{x \to -\infty} (\omega_1) = \lim_{x \to -\infty} (\omega_2),$$

we might have $H(D^-_{\omega_1}) \neq H(D^-_{\omega_2})$, hence $\Phi^N_n$ might be different for $\omega_1, \omega_2$. However, this would not cause any problem for the definition of the extended tau-function.

Let $C = C_\omega$ and $C' = C_{\omega'}$ with $c > 1$. One can assume $D^+_{\omega'} \subset D^+_{\omega'}$, $D^-_{\omega'} \subset D^-_{\omega'}$. If $N \geq 1$ is an integer, for $u \in \Phi^N_n$ and $b' > b$ (48) an identity

$$\frac{1}{z-\lambda} = \sum_{k=0}^{N-1} (z-b')^{-k-1} (\lambda-b')^k + (z-b')^{-N} \frac{(\lambda-b')^N}{z-\lambda}$$

implies

$$u(z) = \sum_{k=0}^{N-1} a_k (z-b')^{-k-1} + \frac{(z-b')^{-N}}{2\pi i} \int_{C} (\lambda-b')^N (u(\lambda) - f(\lambda)) d\lambda$$

(50)

with

$$a_k = \frac{1}{2\pi i} \int_{C} (\lambda-b')^k (u(\lambda) - f(\lambda)) d\lambda.$$  

(51)

**Lemma 27** There exists a constant $c_1$ such that

$$\left| u(z) - \sum_{k=0}^{N-1} a_k (z-b')^{-k-1} \right| \leq c_1 |z|^{-N+(n-2)/4} \|u\|_{n,N}$$

holds for any $u \in \Phi^N_n$ and $z \in C'$.  

29
Proof. First note that there exists a constant $c_0 > 0$ such that

$$
\int_C \frac{1}{|z - \lambda|^2} |d\lambda| \leq c_0 |\text{Im} z|^{-1}
$$

holds for any $z \in C'$, since

$$
\inf_{\lambda \in C} |z - \lambda| \geq (c - 1) |\text{Im} z|
$$

and $C$ is very close to $\mathbb{R}$ at $x = -\infty$. Therefore, we have from (48) and (52)

$$
|u(z) - \sum_{k=0}^{N-1} a_k (z - b')^{-k-1}|
\leq \frac{|z - b'|^{-N}}{2\pi} \int_C \frac{|\lambda - b'|^N |u(\lambda) - f(\lambda)|}{|z - \lambda|} |d\lambda|
\leq c_1 |z|^{-N} \sqrt{\int_C \frac{1}{|z - \lambda|^2} |d\lambda| \int_C |\lambda|^{2N} |u(\lambda) - f(\lambda)|^2 |d\lambda|}
\leq c_2 |z|^{-N} |\text{Im} z|^{-1/2} \sqrt{\int_C |\lambda|^{2N} |u(\lambda) - f(\lambda)|^2 |d\lambda|}
$$

with

$$
c_2 = \frac{1}{2\pi} \max_{\lambda \in C} \left| \frac{\lambda - b'}{\lambda} \right| \max_{z \in C} \left| \frac{z - b'}{z - b} \right|,
$$

which completes the proof, since $|\text{Im} z| \geq \text{const.} |z|^{-n/2+1}$ if $z \in C'$. \qed

Now, we can introduce a potential class by its Weyl functions on which the KdV flow is constructed. Let $m_{\pm}$ be the Weyl functions of the Schrödinger operator $L_q$ with potential $q$ and set

$$
m(z) = \begin{cases} 
-m_+(-z^2) & \text{Re } z > 0 \\
m_-(-z^2) & \text{Re } z < 0
\end{cases}
$$

and

$$
\mathcal{M}^N_n = \{m; \ m_{\pm}, \ m_o - 1 \in \Phi^N_n \},
$$

for $n \geq 1, N > 0$.

5.2.2 Extension of $\tau$-function

Set a class of $g$ by

$$
\Gamma_n = \{g = e^h; h \text{ is real odd polynomial with degree } \leq n, \text{ and } h(0) = 0\}
$$

for an odd natural number $n$. The argument below also works if we multiply rational functions to $g$, however that makes the notations and the arguments subtle, so we use this definition here. In order to define the $\tau$-function for $m \in \mathcal{M}^N_n$, $g \in \Gamma_n$, set $C = C_{\omega}$, $C' = C_{\omega'}$ with $c > 1$. The integral kernel $N_{g,m}(z, \lambda)$ on $L^2(C)$ and the integral operator on $L^2(C)$ are defined by

$$
\left\{
\begin{array}{ll}
N_{g,m}(z, \lambda) = \frac{1}{2\pi i} \int_C \tilde{g}_o(\lambda')(g \tilde{m})_e(\lambda)+\tilde{g}_e(\lambda')(g \tilde{m})_o(\lambda) (\lambda' - z)(\lambda' - \lambda)^{-1} d\lambda' \\
N_m(g) f(z) = \frac{1}{2\pi i} \int_C N_{g,m}(z, \lambda)f(\lambda) d\lambda
\end{array}\right.
$$

(53)
as we did in (39). The number $N$ will be specified later, and assume to be sufficiently large for the time being. Since $m_\varepsilon, m_\alpha - 1 \in \Phi^N_n$, there exist $f_1, f_2 \in H(D^+)$ such that

$$\int_C \left| z \right|^{2N} |m_\varepsilon(z) - f_1(z)|^2 \, dz < \infty, \quad \int_C \left| z \right|^{2N} |m_\alpha(z) - 1 - f_2(z)|^2 \, dz < \infty$$

are satisfied, hence we set $\tilde{m}_\varepsilon = m_\varepsilon - f_1, \tilde{m}_\alpha = m_\alpha - 1 - f_2$ with these $f_1, f_2$. The estimate of the kernel $N_{g,m}(z, \lambda)$ is fundamental for the main result. Let

$$|g|_C = \max_{z \in C} |g(z)|.$$

Since

$$m_\alpha(z) = -\frac{m_+(z) + m_-(z)}{2\sqrt{z}}$$

and $m_\pm$ are of Herglotz, we see $m_\alpha(z) \neq 0$ on $\mathbb{C}\setminus(-\infty, -\lambda_0]$, hence this together with Lemma 27 implies

$$\min_{\lambda' \in C} |m_\alpha(\lambda')| > 0,$$

if $N > (n - 2)/4$.

**Lemma 28** Suppose $N > (n - 2)/4$. Then, there exists a constant $c_2$ such that

$$|N_{g,m}(z, \lambda)| \leq c_2 |g|_C |\tilde{g}|_{C'} \left( \min_{\lambda' \in C} |m_\alpha(\lambda')| \right)^{-1} (|\tilde{m}_\varepsilon(\lambda)| + |\tilde{m}_\alpha(\lambda)|) \left( |z| |\lambda| \right)^{(n-2)/4}$$

holds for $z, \lambda \in C$.

**Proof.** From (53)

$$|N_{g,m}(z, \lambda)| \leq c_3 |g|_C |\tilde{g}|_{C'} \left( \min_{\lambda' \in C} |m_\alpha(\lambda')| \right)^{-1} \int_{C'} \frac{|\tilde{m}_\varepsilon(\lambda)| + |\tilde{m}_\alpha(\lambda)|}{|\lambda' - z| |\lambda' - \lambda|^n} \, d\lambda'$$

(54)

follows. An estimate similar to (52) holds, namely

$$\int_{C'} |\lambda' - z|^{-2} |d\lambda'| \leq c_4 |\text{Im } z|^{-1} \leq c_5 |z|^{n/2-1}$$

if $z \in C$, hence applying Schwarz inequality to (54) concludes the proof. $lacksquare$

Since sufficient conditions for trace class operators are involved, we replace the det by $\det_2$ which is defined for Hilbert-Schmidt class operators, namely

$$\det_2 (I + A) = e^{-\text{tr} A} \det (I + A).$$

Moreover, the kernel $N_{g,m}(z, \lambda)$ does not decay with respect to $z$, hence it should be changed so that the resulting operator $N_m(g)$ turns to be of Hilbert-Schmidt class. Fortunately we have the following property of determinants:

$$\det (I + A) = \det (I + H^{-1} AH)$$

for any operator $H$. So for a suitable $M > 0$ set

$$H(z) = z^M,$$
and replace \( N_{g,m} (z, \lambda) \) with a kernel
\[
\tilde{N}_{g,m} (z, \lambda) = H(z)^{-1} N_{g,m} (z, \lambda) H(\lambda).
\]

The necessary sufficient condition for the new operator \( \tilde{N}_m (g) \) to be a Hilbert-Schmidt class operator is
\[
\int_{C^2} \left| \tilde{N}_{g,m} (z, \lambda) \right|^2 |dz| |d\lambda| < \infty. \tag{55}
\]

We replace the part \( \text{tr} A \) in the definition of \( \det \) by the integral
\[
\int_{C} \tilde{N}_{g,m} (z, z) dz = \int_{C} N_{g,m} (z, z) dz \tag{56}
\]
regardless of the tracability of \( \tilde{N}_m (g) \). Then we require for \( N_{g,m} \) to satisfy
\[
\int_{C} |N_{g,m} (z, z)| |dz| < \infty. \tag{57}
\]

If the kernel \( \tilde{N}_{g,m} (z, \lambda) \) satisfies (55) and (57), the \( \tau \)-function is defined by
\[
\tau_m (g) = \exp \left( \int_{C} N_{g,m} (z, z) dz \right) \det_2 \left( I + \tilde{N}_m (g) \right). \tag{58}
\]

This determinant and trace was used by [29]. A sufficient condition for (55) follows from the estimate (54) and it is
\[
N \geq M + (n - 2) / 4 \text{ and } (n - 2) / 2 - 2M < -1. \tag{59}
\]

\( M \) can be chosen freely if it satisfies \( n/4 < M \leq N - (n - 2) / 4 \), which is possible if \( N > (n - 1) / 2 \). Since
\[
(\left| |\tilde{m}_c (\lambda)| + |\tilde{m}_o (\lambda)| \right| |\lambda|^{(n-2)/2} = |\lambda|^{N - (\left| |\tilde{m}_c (\lambda)| + |\tilde{m}_o (\lambda)| \right|) |\lambda|^{-N+(n-2)/2}}
\]
(57) is satisfied if \(-2N + n - 2 < -1\), which is equivalent to \( N > (n - 1) / 2 \). Consequently we have

**Lemma 29:** Assume \( N > (n - 1) / 2 \). Then, for \( M \) satisfying \( n/4 < M \leq N - (n - 2) / 4 \) (55) and (57) are satisfied, and the \( \tau \)-function \( \tau_m (g) \) can be defined by (58) for \( g \in \Gamma_n, m \in \mathcal{M}_n^N \). Moreover, so defined \( \tau_m (g) \) is continuous on \( \mathcal{M}_n^N \).

**Proof.** We have only to show the continuity, which require us to prove the continuity of the Hilbert-Schmidt norm of \( \tilde{N}_m (g) \) and the integral (56) with respect to \( m \). We prove only the continuity of \( \min_{\lambda \in C^1} |m_o (\lambda')| \), since the other quantities are easy to treat. Let \( m_1, m_2 \in \mathcal{M}_n^N \). Applying Lemma 27 to \( m_{1,o} - m_{2,o} \in \Phi_n^N \) yields
\[
\left| m_{1,o} (z) - m_{2,o} (z) - \sum_{k=0}^{N-1} a_k (z - b')^{-k-1} \right| \leq c_1 |z|^{-N+(n-2)/4} \| m_{1,o} - m_{2,o} \|_{n,N} \tag{60}
\]
Then, we have
\[ a_k = \frac{1}{2\pi i} \int_C (\lambda - b')^k \left( m_{1,0} - m_{2,0} - f(\lambda) \right) d\lambda, \]
where \( f(z) = f_1(z) - f_2(z) \in H(D^+_z) \) and \( f_j \in H(D^+_z) \) are such that
\[ \lambda^N (m_j - f_j) \in L^2(C_\omega). \]

Proof. Take an \( \lambda \in C \)

Lemma 30
The \( \tau \)-function defined by (58) for \( g \in I_n \) coincides with the \( \tau \)-function defined by \( N_m(g) \) in (39) if \( m \in \mathcal{M}_{\sqrt{\tau}}^{refl} \).

Proof. Take an \( m \in \mathcal{M}_{\sqrt{\tau}}^{refl} \). For simplicity we use \( \det \) in (58) instead of \( \det_2 \). We divide the curve \( C \) defining \( \tilde{N}_m(g) \) into \( C_1, C_2 \), where \( C_1 \) contains the interval \([-r, r]\) and is bounded, and \( C_2 \) is the rest of the curve \( C \) extending to \(-\infty\). Notice that \( m_c, m_o \) have singularities only on \([-r, r]\), hence \( C_2 \) has no singularities of \( m_c, m_o \) in its inside. And let \( N_1 \) and \( N_2 \) be the operators obtained by restricting \( \tilde{N}_m(g) \) on \( C_1 \) and \( C_2 \) respectively. Then, the analyticity of \( m_c, m_o \) in the inside of \( C_2 \) implies
\[ N_2 N_1 = N_2^2 = 0. \]

Then, we have
\[ \det \left( I + \tilde{N}_m(g) \right) = \det (I + N_1 + N_2) = \det(I + N_1), \]
by which we can identify the two \( \tau \)-functions if \( m \in \mathcal{M}_{\sqrt{\tau}}^{refl} \).

Our goal is to show that \( (K(g)q)(x) = -2\partial_x^2 \log \tau_m(qe_x) \) defines a flow and \( (K(g)q)(x) \) gives a solution to the KdV equation for \( g_1(z) = e^{-4z^2} \). To accomplish it we need to approximate \( m \in \mathcal{M}_n^N \) by elements of \( \mathcal{M}_{\sqrt{\tau}}^{refl} \), and show \( \tau_m(g) \) is larger than \( 0 \). For that purpose we need to show a modified cocycle property of \( \tau_m(g) \). The cocycle property also is proved by the approximation. Therefore, our first task is to show the approximation.
5.2.3 Approximation of \( m \in M_n^N \) by \( M_{\lambda_0, \lambda_1}^{refl} \)

Our goal is to show that \( u(t, x) = -2\partial_x^2 \log \tau_m \left( e^{xz + tz^2} \right) \) satisfies the KdV equation for \( m \in M_n^N \). Although a direct proof using the definition of the extended \( \tau_m \) is preferable, it seems difficult to proceed this way. The approach we take here is to make use of some approximation of \( m \) by a sequence of reflectionless \( m \), since for reflectionless \( m \) we know already the associated \( u(t, x) \) satisfies the KdV equation. We use this approximation also to prove the property \( m(g) \neq 0 \).

For \( 0 < \lambda_0 < \lambda_1 \) denote by \( M_{\lambda_0, \lambda_1}^{refl} \) the set

\[
M_{\lambda_0, \lambda_1}^{refl} = \left\{ m; \text{the associated } L_q \text{ has its spectrum in } [\lambda_0, \infty) \right\}.
\]

If \( \lambda > \lambda_1, -\lambda_0 \), then \( M_{\lambda_0, \lambda_1}^{refl} \subset M_{\lambda_0, \lambda_1}^{refl} \) holds. The approximation of \( m \in M_n^N \) by elements of \( \bigcup_{\lambda_0 > 0} M_{\lambda_0, \lambda_1}^{refl} \) in the space \( M_n^N \) is technically very much involved so the detailed proof will be given in Appendix, and in this section we provide a summary of it.

**Lemma 31** Let \( m \in M_n^N \) and \( N' \) be a positive integer satisfying \( N' < (n/4 - 1) \wedge N \). Then, for any \( \epsilon > 0 \) there exist \( \lambda_1 > 0 \) and \( m' \in M_{\lambda_0, \lambda_1}^{refl} \) such that

\[
\| m - m' \|_{n, N'} + \| m_0 - m'_0 \|_{n, N'} < \epsilon
\]

holds.

The proof consists of two steps.

**1. Projection to \( M_{\lambda_0, \lambda_1}^{refl} \)** For \( a < 0 < b \) let \( C^{a, b} \) be a simple closed smooth curve cutting \( \mathbb{R} \) at \( a, b \).

\[ C^{a, b} \]

For two Herglotz functions \( m_\pm \) form \( m \) by (38) and set

\[
\begin{align*}
m_{e}^{a, b}(z) &= \frac{1}{2\pi i} \int_{C^{a, b}} \frac{m_e(\lambda)}{z - \lambda} d\lambda \\
m_{o}^{a, b}(z) &= 1 + \frac{\sqrt{1 - a/z}}{2\pi i} \int_{C^{a, b}} \frac{m_o(\lambda) - 1}{(z - \lambda) \sqrt{1 - a/\lambda}} d\lambda,
\end{align*}
\]

and

\[
\begin{align*}
m_+^{a, b}(z) &= -m_{e}^{a, b}(-z) - \sqrt{-z} m_{o}^{a, b}(-z) \\
m_-^{a, b}(z) &= m_{o}^{a, b}(-z) - \sqrt{-z} m_{e}^{a, b}(-z).
\end{align*}
\]

Then, one can show \( m^{a, b}(z) = m_{e}^{a, b}(z^2) + zm_{o}^{a, b}(z^2) \in M_{-b, -a}^{refl} \). By deforming the curve \( C^{a, b} \) fixing the cutting points \( a, b \) one can see easily that \( m_{e}^{a, b} \) and \( m_{o}^{a, b} \) are analytic on \( \mathbb{C} \setminus [a, b] \). The point here is to verify \( m_{e_0}^{a, b} \) are of Herglotz.
2. Proof of $m^{a,b} \rightarrow m$ when $a \rightarrow -\infty$

Let $C^1_a$, $C^2_a$ be the curve and the line below and set $C_a = C^1_a + C^2_a$.

Then, the differences $m_e - m^{a,b}_e$ and $m_o - m^{a,b}_o$ can be expressed by integrals on $C_a$, which are divided into two integrals on $C^1_a$ and $C^2_a$. The convergence of the integral on $C^2_a$ can be shown without difficulties, however, the integral on $C^2_a$ becomes an obstacle to have the convergence in $\mathcal{M}_n^N$, which restricts the convergence at most in $\mathcal{M}_n^{N'}$ for $N'$ such that $N' < n/4 - 1$.

5.2.4 Positivity of $\tau_m(g)$

To avoid the singularity of the flow which will be defined later the non-degeneracy of $\tau_m(g)$ is necessary. This property is shown by following the proof for $m$ which is reflectionless on a right half line of $\mathbb{R}$. For this purpose we need to extend the action (Lemma 26) of $\Gamma_n$ to $\mathcal{M}_n^N$ and the cocycle property (Lemma 25).

Extension of $\Gamma_n$-action

Recall

$$q_\zeta(z) = (1 - z\zeta^{-1})^{-1}.$$}

For $W \in Gr^{(2)}$ we have defined $m_W(z)$ and obtained a formula

$$\tau_W(q_{\zeta_1}, q_{\zeta_2}) = (1 + \varphi_W(\zeta_1))(1 + \varphi_W(\zeta_2)) \frac{m_W(\zeta_1) - m_W(\zeta_2)}{\zeta_1 - \zeta_2}$$

$$= \tau_W(q_{\zeta_1}) \tau_W(q_{\zeta_2}) \frac{m_W(\zeta_1) - m_W(\zeta_2)}{\zeta_1 - \zeta_2}$$

in (51) of Appendix of ([20]), which together with the property

$$m_W(z) = z + O(z^{-1})$$

yields

$$m_W(z) = z + \int_{-\infty}^{z} \frac{\tau_W(q^2_\zeta) - 1}{\tau_W(q_\zeta^2)} d\zeta.$$
Lemma 27 shows that there exists a rational function which causes difficulty to have finite

\[ \left( q^2 \right)_e (z) = 1 + z\zeta^{-2}, \quad \left( q^2 \right)_o (z) = -\zeta^{-1}, \]

we have

\[
\begin{cases}
\left( gq^2 \right)_o (\lambda') = (1 + \lambda'\zeta^{-2}) \hat{g}_o (\lambda') - \zeta^{-1}\hat{g}_e (\lambda') \\
\left( gq^2 \right)_e (\lambda') = (1 + \lambda'\zeta^{-2}) \hat{g}_e (\lambda') - \zeta^{-1}\hat{g}_o (\lambda')
\end{cases}
\]

which causes difficulty to have finite \( N_{gq^2, m}(z, \lambda) \) in the expression (53). To avoid this apparent divergence we modify \( N_{g, m}(z, \lambda) \). Since \( m_\circ (z) - 1 \in \Phi^N_n \), Lemma 27 shows that there exists a rational function \( f_0 \) such that \( f_0(z) = O(z^{-1}) \) and

\[ |m_\circ (z) - 1 - f_0(z)| \leq c_1 |z|^{-N+(n-2)/4} \|m_\circ - 1\|_{n, N}. \]

Let \( N_1 \) be the greatest integer such that \( N_1 \leq N - (n-2)/4 \) and set

\[ f(z) = \sum_{k=1}^{N_1} (-f_0(z))^k. \]

Then, we have

\[ m_\circ (z)^{-1} = 1 + f(z) + O \left( |z|^{-N+(n-2)/4} \right) \text{ for } z \in C'. \] (62)

Since one can assume that \( f \) has no poles in the inside of \( C' \), we have

\[ N_{g, m}(z, \lambda) = N^1_{g, m}(z, \lambda) + (1 + f(z)) N^2_{g, m}(z, \lambda) + \frac{f(z) - f(\lambda)}{\lambda - z} \tilde{m}_\circ (\lambda), \] (63)

where

\[
\begin{cases}
N^1_{g, m}(z, \lambda) = \frac{1}{2\pi i} \int_{C'} \frac{M_{g, m}(\lambda', \lambda) \left( m_\circ (\lambda')^{-1} - 1 - f(\lambda') \right)}{(\lambda' - z)(\lambda - \lambda')} d\lambda' \\
N^2_{g, m}(z, \lambda) = \frac{M_{g, m}(z, \lambda) - \tilde{m}_\circ (\lambda)}{\lambda - z}
\end{cases}
\]

with

\[ M_{g, m}(z, \lambda) = \hat{g}_o (z) (g\tilde{m})_e (\lambda) + \hat{g}_e (z) (g\tilde{m})_o (\lambda). \]

Set

\[ \rho (\zeta) = \inf_{\lambda \in C} |1 - \lambda\zeta^{-1}|. \] (65)

**Lemma 32** Assume \(-N+(n-2)/4+1 \leq -1\). Then, the kernels \( N^1_{gq^2, m}(z, \lambda) \), \( N^2_{gq^2, m}(z, \lambda) \) for \( j = 1, 2 \) have expansions

\[
\begin{align*}
N^1_{gq^2, m}(z, \lambda) &= \sum_{k=0}^{2L-1} N^1_k (z, \lambda) \zeta^{-k} + \zeta^{-2L} \Delta^1_k (z, \lambda) \\
N^2_{gq^2, m}(z, \lambda) &= \sum_{k=0}^{2L-1} N^2_k (z, \lambda) \zeta^{-k} + \zeta^{-2L} \Delta^2_k (z, \lambda) \\
N^1_{gq^2, m}(z, \lambda) &= \sum_{k=0}^{2L-1} \tilde{N}^1_k (z, \lambda) \zeta^{-k} + \zeta^{-2L} \tilde{\Delta}^1_k (z, \lambda) \\
N^2_{gq^2, m}(z, \lambda) &= \sum_{k=0}^{2L-1} \tilde{N}^2_k (z, \lambda) \zeta^{-k} + \zeta^{-2L} \tilde{\Delta}^2_k (z, \lambda)
\end{align*}
\]
with estimates

\[
\begin{align*}
\left| N^1_k (z, \lambda) \right| & \le c_1 |\lambda|^{L+(n-2)/4} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |z|^{(n-2)/4} \\
\left| \Delta^1_k (z, \lambda) \right| & \le c_1 \rho (\zeta^2) |\lambda|^{L+(n-2)/4} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |z|^{(n-2)/4} \\
\left| N^2_k (z, \lambda) \right| & \le c_1 \left| \frac{\sqrt{\nu} e^h (\sqrt{q}) - \sqrt{\nu} e^h (\sqrt{\lambda})}{z - \lambda} \right| |\lambda|^L (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) \\
\left| \Delta^2_k (z, \lambda) \right| & \le c_1 \rho (\zeta^2) \left| \frac{\sqrt{\nu} e^h (\sqrt{q}) - \sqrt{\nu} e^h (\sqrt{\lambda})}{z - \lambda} \right| |\lambda|^L (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|)
\end{align*}
\]  

(66)

and

\[
\begin{align*}
\tilde{N}^1_k (z, \lambda) & \le c_1 |\lambda|^{L+(n-2)/4} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |z|^{(n-2)/4} \\
\tilde{\Delta}^1_k (z, \lambda) & \le c_1 \rho (\zeta^2)^2 |\lambda|^{L+(n-2)/4} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |z|^{(n-2)/4} \\
\tilde{N}^2_k (z, \lambda) & \le c_1 \left| \frac{\nu e^{h (\sqrt{q})} - \nu e^{h (\sqrt{\lambda})}}{z - \lambda} \right| |\lambda|^L (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) \\
\tilde{\Delta}^2_k (z, \lambda) & \le c_1 \rho (\zeta^2)^2 \left| \frac{\nu e^{h (\sqrt{q})} - \nu e^{h (\sqrt{\lambda})}}{z - \lambda} \right| |\lambda|^L (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|)
\end{align*}
\]  

(67)

for \( k = 0, 1, \ldots, 2L - 1 \) and \( z, \lambda \in C \) with some constant \( c_1 \).

**Proof.** \( q_\zeta \) has an expansion

\[
q_\zeta (z) = \sum_{k=0}^{2L-1} z^k \zeta^{-k} + z^{2L} \zeta^{-2L} q_\zeta (z)
\]

for a positive integer \( L \), which allows us to expand \( M_{gq, m} (z, \lambda) \) as

\[
M_{gq, m} (z, \lambda) = \sum_{k=0}^{2L-1} M^1_k (z, \lambda) \zeta^{-k} + \delta^1_k (z, \lambda) \zeta^{-2L},
\]

where

\[
\begin{align*}
M^1_k (z, \lambda) &= \tilde{g}_\zeta (z) \left( g \lambda^k \tilde{m} \right)_e (\lambda) + \tilde{g}_\zeta (z) \left( g \lambda^k \tilde{m} \right)_o (\lambda) \\
&\quad - \left( z \tilde{g}_\zeta (z) \left( g \lambda^{k-1} \tilde{m} \right)_o (\lambda) + \tilde{g}_\zeta (z) \lambda^k (g \lambda^{k-1} \tilde{m})_o (\lambda) \right) \\
\delta^1_k (z, \lambda) &= \tilde{g}_\zeta (z) \lambda^k (g \lambda^{k-1} \tilde{m})_e (\lambda) + \tilde{g}_\zeta (z) \lambda^k (g \lambda^{k-1} \tilde{m})_o (\lambda) \\
&\quad - \left( \tilde{g}_\zeta (z) \lambda^{k-1} (g \lambda^k \tilde{m})_e (\lambda) + \tilde{g}_\zeta (z) \lambda^{k-1} (g \lambda^k \tilde{m})_o (\lambda) \right) \zeta^{-1}
\end{align*}
\]

Set

\[
\begin{align*}
N^1_k (z, \lambda) &= \frac{1}{2\pi i} \int_{C'} M^1_k (\lambda', \lambda) \left( m_o (\lambda')^{-1} - f (\lambda') \right) d\lambda' \\
\Delta^1_k (z, \lambda) &= \frac{1}{2\pi i} \int_{C'} \delta^1_k (\lambda', \lambda) \left( m_o (\lambda')^{-1} - f (\lambda') \right) d\lambda'
\end{align*}
\]

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Then, if \(-N+(n-2)/4+1/2 \leq -1\), (62) shows that \(\lambda' \hat{g}_o (\lambda') \left( m_o (\lambda')^{-1} - f (\lambda') \right)\)
is bounded and (66) for \(N_k^j (z, \lambda)\) and \(\Delta_k^j (z, \lambda)\) are valid. Here we have used the estimate in the proof of Lemma 28 that

\[
\int_{C'} \frac{1}{|\lambda' - z|^2} |d\lambda'| \leq c_2 |\text{Im } z|^{-1} \leq c_3 |z|^{(n-2)/2}
\]

for \(z \in C\).

On the other hand, to estimate the kernel \(N_{g^2, m}^j (z, \lambda)\) note

\[
M_{g, m}^2(z, \lambda) - \tilde{m}_o (\lambda) = \sum_{k=0}^{2L-1} M_k^2(z, \lambda) \zeta^{-k} + \delta_k^2(z, \lambda) \zeta^{-2M}
\]

with

\[
\begin{align*}
M_k^2(z, \lambda) &= (\hat{g}_o (z) - \hat{g}_e (z)) \left( g^{\lambda^j \tilde{m}_o} \right)_e (\lambda) + (\hat{g}_e (z) - \hat{g}_e (\lambda)) \left( g^{\lambda^j \tilde{m}_o} \right)_o (\lambda) - (z \hat{g}_o (z) - \tilde{g}_o (\lambda)) \left( g^{\lambda^j \tilde{m}_o} \right)_e (\lambda) + (\hat{g}_e (z) - \tilde{g}_o (\lambda)) \left( g^{\lambda^j \tilde{m}_o} \right)_o (\lambda) \\
\delta_k^2(z, \lambda) &= \left( \hat{g}_o (z) - \hat{g}_e (\lambda) \right) \lambda^L (q_{g^2} g_{m})_e (\lambda) + (\hat{g}_e (z) - \tilde{g}_o (\lambda)) \lambda^L (q_{g^2} g_{m})_o (\lambda) - \left( z \hat{g}_o (z) - \tilde{g}_o (\lambda) \right) \lambda^L (q_{g^2} g_{m})_e (\lambda) + (\hat{g}_e (z) - \hat{g}_e (\lambda)) \lambda^L (q_{g^2} g_{m})_o (\lambda)
\end{align*}
\]

where we understand \(g \lambda^{1-1} \tilde{m} = 0\). Therefore, we have

\[
N_k^2(z, \lambda) = \frac{M_k^2(z, \lambda)}{\lambda - z}, \quad \Delta_k^2(z, \lambda) = \frac{\delta_k^2(z, \lambda)}{\lambda - z}
\]

and they have the estimates in (66) for \(N_k^2(z, \lambda)\) and \(\Delta_k^2(z, \lambda)\).

The estimates (66) for \(N_{g^2, m}^j (z, \lambda)\) are obtained similarly if we notices an identity

\[
g_{\lambda}^2(z)^2 = \sum_{k=0}^{2L-1} (k+1) \zeta^{-k} z^k + \zeta^{-2L} z^{2L} (2L + q_{\hat{g}}(z)) q_{\hat{g}}(z).
\]

Then

\[
M_{g^2, m}^j(z, \lambda) = \sum_{k=0}^{2L-1} M_k^1(z, \lambda) \zeta^{-k} + \tilde{\delta}_k^1(z, \lambda) \zeta^{-2L}
\]

holds with

\[
\begin{align*}
\tilde{M}_k^1(z, \lambda) &= (k+1) \left( \hat{g}_o (z) \left( g^{\lambda^j \tilde{m}_o} \right)_e (\lambda) + \hat{g}_e (z) \left( g^{\lambda^j \tilde{m}_o} \right)_o (\lambda) \right) - k \left( z \hat{g}_o (z) \left( g^{\lambda^j \tilde{m}_o} \right)_e (\lambda) + \hat{g}_e (z) \left( g^{\lambda^j \tilde{m}_o} \right)_o (\lambda) \right) + (k-1) \left( \tilde{g}_o (z) \left( \lambda^{j-2} g_{m} \right)_e (\lambda) + \tilde{g}_o (z) \left( \lambda^{j-2} g_{m} \right)_o (\lambda) \right)
\end{align*}
\]

\[
\tilde{\delta}_k^1(z, \lambda) = \lambda^L \left( \hat{g}_o (z) \left( (2L + q_{\hat{g}} g_{m})_e (\lambda) + \hat{g}_e (z) \left( (2L + q_{\hat{g}} g_{m})_o (\lambda) \right) - \lambda^L \left( \hat{g}_e (z) \left( (2L + q_{\hat{g}} g_{m})_e (\lambda) + z \hat{g}_o (z) \left( (2L + q_{\hat{g}} g_{m})_o (\lambda) \right) - z \lambda^L \left( \hat{g}_o (z) \left( (2L + q_{\hat{g}} g_{m})_e (\lambda) + \hat{g}_e (z) \left( (2L + q_{\hat{g}} g_{m})_o (\lambda) \right) \right) \zeta^{-1} - 2L \lambda^L \left( z \hat{g}_o (z) \left( (2L + q_{\hat{g}} g_{m})_e (\lambda) + \hat{g}_e (z) \left( (2L + q_{\hat{g}} g_{m})_o (\lambda) \right) \right) \zeta^{-2} + (2L - 1) z \lambda^{j-1} \left( \hat{g}_o (z) \left( g_{m} \right)_e (\lambda) + \hat{g}_e (z) \left( g_{m} \right)_o (\lambda) \right) \zeta^{-1}
\end{align*}
\]
and
\[
\begin{cases}
\tilde{N}_k^1 (z, \lambda) = \frac{1}{2\pi i} \int_{C'} \frac{\tilde{M}_k^2 (\lambda', \lambda) \left( m_o (\lambda')^{-1} - f (\lambda') \right)}{(\lambda' - z) (\lambda - \lambda')} d\lambda', \\
\hat{\Delta}_k^1 (z, \lambda) = \frac{1}{2\pi i} \int_{C'} \frac{\delta_\xi (\lambda', \lambda) \left( m_o (\lambda')^{-1} - f (\lambda') \right)}{(\lambda' - z) (\lambda - \lambda')} d\lambda'.
\end{cases}
\]

In this case, if \(-N + (n - 2)/4 + 1 \leq -1\), then \(\lambda' \tilde{g}_e (\lambda') \left( m_o (\lambda')^{-1} - f (\lambda') \right)\)
is bounded and (66) for \(\tilde{N}_k^1 (z, \lambda)\) and \(\hat{\Delta}_k^1 (z, \lambda)\) are valid.

On the other hand, we have
\[
M_{gq^2, m}(z, \lambda) - \tilde{m}_o (\lambda) = \sum_{k=0}^{2L-1} \tilde{M}_k^2 (z, \lambda) \zeta^{-k} + \tilde{\delta}_\xi (z, \lambda) \zeta^{-2M},
\]
and \(\tilde{M}_k^2 (z, \lambda)\) and \(\tilde{\delta}_\xi (z, \lambda)\) are defined from \(\tilde{M}_k^1 (z, \lambda)\) and \(\tilde{\delta}_\xi (z, \lambda)\) by replacing
\(z^j \tilde{g}_e (z)\), \(z^j \tilde{g}_e (z)\) with \(z^j \tilde{g}_e (z) - \lambda^j \tilde{g}_e (\lambda)\), \(z^j \tilde{g}_e (z) - \lambda^j \tilde{g}_o (\lambda)\) respectively, and set
\[
\tilde{N}_k^2 (z, \lambda) = \frac{\tilde{M}_k^2 (z, \lambda)}{\lambda - z}, \quad \hat{\Delta}_k^2 (z, \lambda) = \frac{\tilde{\delta}_\xi (z, \lambda)}{\lambda - z}.
\]

Then, the estimates (67) are obtained similarly.

The above lemma makes it possible to have asymptotic expansion of \(\tau_m (gq^2)\), \(\tau_m \left( gq^2 \xi \right)\).

**Lemma 33** Suppose \(g \in \Gamma_n\) and \(m \in \mathcal{M}_n^N\) with \(N > (n + 1)/2\). Then, \(\tau_m (gq\xi)\) and \(\tau_m \left( gq^2 \xi \right)\) are analytic on \(\mathbb{C} \setminus \left( [-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] \times i\mathbb{R} \right)\), and satisfy
\[
\begin{cases}
\tau_m (gq\xi) = F_1 (\zeta^{-1}) + \delta_1 (\zeta) \\
\tau_m \left( gq^2 \xi \right) = F_2 (\zeta^{-1}) + \delta_2 (\zeta)
\end{cases}
\]

where \(F_1, F_2\) are entire functions satisfying
\[
\begin{cases}
|\delta_1 (\zeta)| \leq c_1 |\zeta^2|^{-L} \rho (\zeta^{-2})^{-1} \\
|\delta_2 (\zeta)| \leq c_1 |\zeta^2|^{-L} \rho (\zeta^{-2})^{-2}
\end{cases}
\]
with an integer \(L = N - (n + 3)/2\). Moreover, \(F_1 (0) = F_2 (0) = \tau_m (g)\), \(F_2 (0) = 2F_1' (0)\) are valid.

**Proof.** Let \(H (z) = z^M\) for \(M > 0\) and examine the Hilbert-Schmidt condition and the integrability of \(H^{-1} N_m (gq\xi) H\). The estimates (66) show the square of the Hilbert-Schmidt norm of \(H^{-1} N_m (gq\xi) H\) is dominated by
\[
\int_{\mathbb{C}^2} \left| |H|^{M+L+(n-2)/4} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) \right|^{(n-2)/4-M} |dz| |d\lambda|,
\]
which is finite if
\[M + L + (n - 2)/4 \leq N, \quad (n - 2)/4 - M < -1/2.\]
This condition is equivalent to
\[(n-2)/4 + 1/2 < M \leq N - L - (n-2)/4,\]
which is achieved if
\[L < N - (n-2)/4 - n/4 = N - (n-1)/2.\]

On the other hand, the square of the Hilbert-Schmidt norm of \(H^{-1} (1 + f(z)) N_m^2 (gq_k^2) H\) is dominated by
\[\int_{C^2} |dz| |d\lambda| \left( \frac{\sqrt{z} e^{b(\sqrt{z})} - \sqrt{\lambda} e^{b(\sqrt{\lambda})}}{z - \lambda} \right) |\lambda|^{L+M} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |z|^{-M} \right)^2.\]

Lemma 56 shows that this is finite if
\[2(L+M) + n - 2M \leq 2N, \quad M \leq (n+1)/2\]
or equivalently if \(L \leq N - n/2\). In (63) the square of the Hilbert-Schmidt norm of \(H(z)^{-1} (f(z) - f(\lambda)) \tilde{m}_o (\lambda) H(\lambda) / (\lambda - z)\) is dominated by
\[\int_{C^2} |dz| |d\lambda| \left| \frac{f(z) - f(\lambda)}{\lambda - z} \right|^2 |\lambda|^{2M} |z|^{-2M} |\tilde{m}_o (\lambda)|^2.\]

Since \(f(z)\) is a rational function satisfying \(O(z^{-1})\), the above integral is dominated by
\[\int_{C^2} |dz| |d\lambda| |\lambda|^{2(M-1)} |z|^{-2(M+1)} |\tilde{m}_o (\lambda)|^2,\]
which is finite if \(2(M + 1) > 1\) and \(M - 1 \leq N\). Consequently, \(H^{-1} N_m (gq_k) H\) is of Hilbert-Schmidt class if \(L \leq N - n/2\).

Similarly, the estimates (67) shows \(H^{-1} N_m (gq_k^2) H\) is of Hilbert-Schmidt class if \(L \leq N - (n+1)/2\). In this case \(M\) is an arbitrary number satisfying \(n/4 < M \leq N - L - (n-2)/4\).

The integral \(\int_C |N_{\zeta \beta, m} (\lambda, \lambda) ||d\lambda|\) is finite if
\[\int_C |\lambda|^{L+(n-2)/4+(n-2)/4} (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |d\lambda| + \int_C \sqrt{\lambda} h' (\sqrt{\lambda}) |\lambda|^L (|\tilde{m}_e (\lambda)| + |\tilde{m}_o (\lambda)|) |d\lambda|

is finite, hence
\[L + (n-2)/2 - N < -1/2, \quad n/2 + L - N < -1/2\]
is sufficient, which is satisfied for \(L = N - (n+3)/2\).

To obtain (68), (69) set
\[A = H^{-1} N_m (gq_k) H, \quad B = \sum_{k=0}^{2L-1} (H^{-1} (N_k^1 + (1 + f) N_k^2) H) \zeta^{-k} + B_0,\]
where $B_0$ is the operator

$$B_0 \triangleq H(z)^{-1} (f(z) - f(\lambda)) \bar{m}_0(\lambda) H(\lambda)/(\lambda - z).$$

Then, due to Theorem 6.5 of [32] the function

$$\delta_1(\zeta) \equiv \det_2(I + A) - \det_2(I + B)$$

is dominated as

$$|\delta_1(\zeta)| \leq \|A - B\|_{H.S.} \exp\left(c(\|A\|_{H.S.} + \|B\|_{H.S.} + 1)^2\right).$$

with a universal constant $c$. Since we have from Lemma 32

$$\begin{cases}
\|A - B\|_{H.S.} = \left\|H^{-1}(1 + f)\left(\Delta_1^2 + \Delta_2^2\right)\right\|_{H.S.} \leq c_1 |\zeta|^2 \rho(\zeta)^{-1}, \\
\|A\|_{H.S.} \leq \|A - B\|_{H.S.} + \|B\|_{H.S.} \leq c_1 |\zeta|^2 \rho(\zeta)^{-1} + c_2,
\end{cases}$$

the first estimate of (69) follows. The second estimate also follows similarly. Since $\det_2(I + z_1A_1 + \cdots + z_mA_m)$ is an entire function on $\mathbb{C}^n$ if $A_j$ are of Hilbert-Schmidt (Theorem 3.3 of [32]), $\det_2(I + B)$ as a function of $\zeta$ takes a form of $F_1(\zeta^{-1})$ with an entire function $F_1$, which is valid also for $\tau_m(gq_0^2)$.

The identities $F_1(0) = F_2(0) = \tau_m(g)$ obey from $q_{\zeta} \to 1$ as $\zeta \to \infty$. On the other hand, in (61) for $m \in \mathcal{M}_n^\tau$ we know $g \cdot m \in \mathcal{M}_n^{\tau}$ and

$$(g \cdot m)'(z) = \frac{\tau_m(g) \tau_m(gq_0^2)}{\tau_m(gq_0^2)}$$

holds. Since $(g \cdot m)'(z) = 1 + O(z^{-2})$, we have the second identity $F_2'(0) = 2F_2(0)$ in this case. The identity for general $m \in \mathcal{M}_n^N$ can be proved by approximation.

Here we use the notation $\mathcal{M}_n^N(C)$ in place of $\mathcal{M}_n^N$ if it is necessary to indicate the dependence on the curve $C$.

**Proposition 34** Assume $n \geq 25$ and $N > n + 3$. Then, the following statements hold for $g, g_1, g_2 \in \Gamma_n$ and $m \in \mathcal{M}_n^N(C)$.

(i) If $\tau_m(g) > 0$, then $g \cdot m$ can be defined as an extension of Lemma 26 by

$$(g \cdot m)(z) = z + \int_0^z \frac{\tau_m(g) \tau_m(gq_0^2)}{\tau_m(gq_0^2)} - 1 \, d\zeta,$$

and $g \cdot m \in \mathcal{M}_n^{N-n-3}(C')$. The correspondence $\mathcal{M}_n^N(C) \ni m \mapsto g \cdot m \in \mathcal{M}_n^{N-n-3}(C')$ is continuous.

(ii) Assume $\tau_m(g_1) > 0$. Then, an identity

$$\tau_m(g_1g_2) = \tau_m(g_1)\tau_{g_1}(g_2) \exp\left(\frac{1}{2\pi i} \int_{C_1} \frac{\tau_m(g_1q_0)}{\tau_m(g_1)} - f(\zeta) \, d\zeta\right)$$

holds as an extension of Lemma 13, where $f \in H(D_+)$ is chosen so that $\zeta^{n-1}(\log(\tau_m(g_1q_0)/\tau_m(g_1)) - f(\zeta)) \in L^1(C_1)$ and $C_1$ is a curve located in the outside of $\sqrt{C'}$ such that $\tau_m(g_1q_0) \neq 0$ on the outside of $C_1$. 41
Proof. Let \( N' = 4 \) and \( L' = 1 \). Then, \( N' < (n/4 - 1) \wedge N \) and \( L' = N' - (n'+3)/2 \) hold with \( n' = 3 \). Let \( m_k \in \mathcal{M}_{N_0, \lambda_1} \) converge to \( m \) in \( \mathcal{M}'_{N'} \) whose existence was proved in Lemma 31. We know that \((g \cdot m_k)\) are analytic on \( \mathbb{C} \setminus \{[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] \cup i\mathbb{R}\} \) and satisfy \( \text{Im} (g \cdot m_k) > 0 \). On the other hand, \( \tau_{m_k} (g q^2) \) converge to \( \tau_m (g q^2) \) compact uniformly on \([[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] \cup i\mathbb{R}] \) due to the continuity of \( \tau_m (g) \) with respect to \( m \in \mathcal{M}'_{N'} \) due to Lemma 29, and Lemma 33 shows that there exist entire functions \( F_{1,k} (z) \), \( F_{2,k} (z) \) and error functions \( \delta_{1,k} (\zeta) \), \( \delta_{2,k} (\zeta) \) such that

\[
\begin{align*}
\tau_{m_k} (g q^2) &= F_{1,k} (\zeta^{-1}) + \delta_{1,k} (\zeta) \\
\tau_{m_k} (g q^2) &= F_{2,k} (\zeta^{-1}) + \delta_{2,k} (\zeta)
\end{align*}
\]

and

\[
\begin{align*}
|\delta_{1,k} (\zeta)| &\leq c_1 |\zeta|^2 \rho (\zeta)^{-1} \\
|\delta_{2,k} (\zeta)| &\leq c_1 |\zeta|^2 \rho (\zeta)^{-2}
\end{align*}
\]

(71)

with a constant \( c_1 \) which is independent of \( k \). Since \( F_{1,k} (z) \), \( F_{2,k} (z) \) converge compact uniformly on \( \mathbb{C} \) to \( F_1 (z) \), \( F_2 (z) \), the formula

\[
(g \cdot m_k) (z) = z + \int_{\infty}^{z} \left( \frac{\tau_{m_k} (g) \tau_{m_k} (g q^2)}{\tau_{m_k} (g q^2)^2} - 1 \right) d\zeta
\]

and (71), (72) show \((g \cdot m_k) (z)\) are uniformly bounded on any compact set of \( \{ |z| > r \} \setminus (\mathbb{R} \cup i\mathbb{R}) \) for an \( r > 0 \), which together with the property \( \text{Im} (g \cdot m_k) (z) \) \( / \) \( \text{Im} z \) \( > \) \( 0 \) establishes the convergence of \((g \cdot m_k) (z)\) to some analytic function on \( \mathbb{C} \setminus \{[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] \cup i\mathbb{R}\} \), which is denoted by \((g \cdot m) (z)\), and it has the property \( \text{Im} (g \cdot m) (z) \) \( / \) \( \text{Im} z \) \( > \) \( 0 \).

The expression

\[
(g \cdot m) (z) = z + \int_{\infty}^{z} \left( \frac{\tau_m (g) \tau_m (g q^2)}{\tau_m (g q^2)^2} - 1 \right) d\zeta
\]

is valid at least for \( z \in D_- \) (outside of \( C \)) if \( |z| \) is sufficiently large. Here we have used the assumption \( \tau_m (g) \neq 0 \), which shows \( \tau_m (g q^2) \neq 0 \) for sufficiently large \( \zeta \) since \( \tau_m (g q^2) \to \tau_m (g) \neq 0 \) as \( \zeta \to \infty \). More precisely we have

\[
\frac{\tau_m (g) \tau_m (g q^2)}{\tau_m (g q^2)^2} = \tau_m (g) F_2 (\zeta^{-1}) F_1 (\zeta^{-1})^{-2} + O \left( |\zeta|^2 \rho (\zeta)^{-2} \right),
\]

where

\[
\tau_m (g) F_2 (\zeta^{-1}) F_1 (\zeta^{-1})^{-2} = 1 + O \left( |\zeta|^2 \right)
\]

holds due to \( F_1 (0) = F_2 (0) = \tau_m (g) \), \( F'_2 (0) = 2 F'_1 (0) \), which makes (74) convergent, and yields estimate for \( z^2 \in C' \)

\[
\left| \int_{\infty}^{z} |\zeta|^2 \rho (\zeta)^{-2} |d\zeta| \right| = \frac{1}{2} \left| \int_{\infty}^{z} |\lambda|^2 \rho (\lambda)^{-2} |d\lambda| \right|
\]

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with $L = N - (n + 3)/2$. Noting

$$\rho(\zeta) = \inf_{\lambda \in \mathcal{C}} |1 - \lambda \zeta^{-1}| = |\zeta|^{-1} \inf_{\lambda \in \mathcal{C}} |\zeta - \lambda| \geq |\zeta|^{-1} |\text{Im} \zeta|$$

we have

$$\left| \int_{\infty}^{z^2} |\lambda|^{-L} \rho(\lambda)^{-2} |\lambda|^{-1/2} d\lambda \right| \leq c_1 \int_{\infty}^{z^2} |\lambda|^{-L+2-1/2} |\text{Im} \lambda|^{-2} |d\lambda|$$

$$= c_2 \int_{x^{-\alpha}}^{\infty} (x + t)^{-L+2-1/2} t^{-2} dt$$

$$= c_2 x^{-L+2-1/2-2+1} \int_{x^{-\alpha}}^{\infty} (1 + s)^{-L+2-1/2} s^{-2} ds$$

$$\leq c_3 x^{-L+3/2+\alpha} = c_3 |z^2|^{-N+n+2}$$

with $z^2 = -x + ci (-x)^{-\alpha} (\alpha = n/2 - 1)$. Therefore, $g \cdot m \in \mathcal{M}_{n}^{N-n-3}$ holds.

The statement (ii) also is shown by approximation of $m$ by $m_k$. (68) shows

$$\frac{\tau_m (g_1 q_{\zeta})}{\tau_m (g_1)} = 1 + \left( \frac{F_1 (\zeta^{-1})}{\tau_m (g_1)} - 1 \right) + \delta_1 (\zeta)$$

with entire $(F_1 (\zeta) / \tau_m (g_1) - 1)$ satisfying $(F_1 (0) / \tau_m (g_1) - 1) = 0$ and $\delta_1 (\zeta) = O \left( |\zeta|^2 |L \rho (\zeta^{-1})| \right)$ which implies (70). ■

**Remark 35** One can show $\mathcal{M}_{n}^{N} \subset \mathcal{M}_{n}^{N'}$ if $n \geq n'$ and $N \geq N'$.\n
\[ \tau_m (g) \] has been defined for $g \in \Gamma_n$, however, this definition can be extended for such a $g$ that $g_e, g_o, \tilde{g_e}, \tilde{g_o}$ are analytic in a neighborhood of $D_+$ and bounded there. Set

$$p_\zeta (z) = q_{-\zeta} (z)^{-1} = 1 + z \zeta^{-1}.$$

**Lemma 36** For $m \in \mathcal{M}_n^N$ and set

$$g (z) = \prod_{j=1}^{l_1} q_{\zeta_j} (z) p_{\zeta_j} (z) \prod_{j=1}^{l_2} q_{\tilde{\zeta}_j} (z) p_{\tilde{\zeta}_j} (z)$$

with $\zeta_j^2 \notin \mathbb{R}$, $\xi_j^2 > -\lambda_0$ ($\lambda_0 = \inf \text{ sp} L_q < 0$). Then, $\tau_m (g) > 0$ holds.

**Proof.** For $m \in \mathcal{M}_{\lambda_0, \lambda_1}^{ref}$ Lemma 36 of [35] shows $\tau_m (g) > 0$. On the other hand Proposition 31 of [35] implies $\tau_m (g)$ is universally described by $\Theta$. Since the positivity of $\tau_m (g)$ has been shown for $m \in \mathcal{M}_{\lambda_0, \lambda_1}^{ref}$ using only the property

\[
\begin{cases}
|\text{Im} m(z)| > 0 \quad \text{for } z \notin \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \\
|\text{Im} m'(x)| \neq 0 \quad \text{for } x \notin \mathbb{R} \setminus [-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] 
\end{cases}
\]

the property $\tau_m (g) > 0$ for $m \in \mathcal{M}_n^N$ can be proved by approximation. ■

**Proposition 37** Suppose $n \geq 25$. For $g = e^h \in \Gamma_n$ and $m \in \mathcal{M}_n^N$ with $N > n + 3$ the tau-function $\tau_m (g)$ satisfies $\tau_m (g) > 0$. \n
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\textbf{Proof.} For any integer \( k \geq 1 \) set
\[ g_k(z) = \left(1 - \frac{h(z)}{2k}\right)^k \frac{1}{1 + \frac{h(z)}{2k}}. \]
Then, \( g_k(-z) = g_k(z)^{-1} \) and \( g_k(z) \to e^{-h(z)} \) as \( k \to \infty \) hold. For \( \alpha > 0 \)
\[ \left|\left(1 - \frac{z}{2k}\right)^k \frac{1}{1 + \frac{z}{2k}}\right| \leq e^{2\alpha} \quad \text{(75)} \]
holds if \( |\text{Re} z| \leq \alpha \) and \( k > \alpha \). Since
\[ h(\sqrt{z}) = \pm i a_1 (-\text{Re} z)^{n/2} + \frac{n}{2} a_1 + o(1) \]
is valid along the curve \( C \) with some \( a_1 \in \mathbb{R} \), we see that \( g_{k,c}, g_{k,o}, \tilde{g}_{k,c}, \tilde{g}_{k,o} \) are analytic in a neighborhood of \( D_+ \) and bounded there. More precisely one can show that there exists a constant \( c \) such that
\[ |g_{k,c}(z)| + |g_{k,o}(z)| + |	ilde{g}_{k,c}(z)| + |	ilde{g}_{k,o}(z)| \leq c \]
is valid for any \( z \in C \) and \( k \geq 1 \). Then, the Hilbert-Schmidt norms of \( N_1(g_{g_k}) - N_1(1) \) and \( \int_C |N_{gg_k,m}(z, z) - N_{1,m}(z, z)| \, dz \) converge to 0, which implies \( \lim_{k \to \infty} \tau_m(g_{gg_k}) = 1 \). Then, (70) shows
\[ \tau_m(g) = \tau_m(g_{gg_k}g_{g_k}^{-1}) = \tau_m(g_{gg_k})\tau_{g_{g_k}}(g_{g_k}^{-1}) \exp \left( \frac{1}{2\pi i} \int_{C_1} \theta(z) \left( \log \frac{\tau_m(g_{gg_k}^z)}{\tau_m(g_{gg_k})} - f(z) \right) \, dz \right) \]
with
\[ \theta(z) = (\log g_{k}^{-1})'(z) = -\frac{h'(z)}{1 - \frac{h(z)}{k}} = \frac{h'(z)}{1 - \frac{h(z)}{k}}. \]
Since \( \tau_m(g_{gg_k}) > 0 \) for sufficiently large \( k \) and \( \tau_{g_{g_k}}(g_{g_k}^{-1}) > 0 \) by Lemma 36, we have \( \tau_m(g) > 0 \). \hfill \blacksquare

\subsection*{5.2.5 Differentiability of \( \tau_m(g_k) \)}

The next theme is to show that \(-2\partial_t^2 \log \tau_m \left( e^{x^2 + tz^2} \right)\) satisfies the KdV equation by establishing the differentiability of \( \tau_m \left( e^{x^2 + tz^2} \right) \).

Recall
\[ M_{g,m}(z, \lambda) = \hat{g}_o(z) (g\tilde{m})_c(\lambda) + \hat{g}_e(z) (g\tilde{m})_o(\lambda) \quad \text{(76)} \]
and (63), (64), namely
\[ N_{g,m}(z, \lambda) = N_{g,m}^1(z, \lambda) + (1 + f(z)) N_{g,m}^2(z, \lambda) + \frac{f(z) - f(\lambda)}{\lambda - z} \tilde{m}_o(\lambda), \]
where

\[
\begin{align*}
N_{g,m}^1(z, \lambda) &= \frac{1}{2\pi i} \int_{C'} M_{g,m}(\lambda', \lambda) \left( m_o(\lambda')^{-1} - 1 - f(\lambda') \right) (\lambda' - z) (\lambda - \lambda')^{-1} \, d\lambda', \\
N_{g,m}^2(z, \lambda) &= \frac{M_{g,m}(z, \lambda) - \tilde{m}_o(\lambda)}{\lambda - z}
\end{align*}
\]

and \(f\) is a rational function such that

\[m_o(z)^{-1} = 1 + f(z) + O\left(|z|^{-N+(n-2)/4}\right)\] for \(z \in C'\).

We consider the differentiability of \(\tau_m(g_t)\) with \(g_t = e^{th}\) for a real odd polynomial \(h\) of degree \(n\). The term depending on \(g_t\) in the kernel \(N_{g,m}(z, \lambda)\) comes from

\[M_{g,m}(z, \lambda) = \hat{g}_{t,o}(z) (g_t \tilde{m})_c(\lambda) + \hat{g}_{t,e}(z) (g_t \tilde{m})_e(\lambda),\]

and it is multiplied by \(h(\sqrt{z})\) if \(\hat{g}_{t,o}(z), \hat{g}_{t,e}(z)\) are differentiated by \(t\) and by \(h(\sqrt{z})\) if \(g_t\) is differentiated. Therefore, \(\partial_t \tilde{N}_m(g_t) = H^{-1} \partial_t N_m(g_t)H\) is of Hilbert-Schmidt for \(H(z) = z^M\), if \(N\) is sufficiently large.

**Lemma 38** If \(m \in M^N_n\) for sufficiently large \(N\), then \(\tau_m(g_t)\) is differentiable with respect to \(t\).

### 5.2.6 KdV flow on \(Q\)

So far we have not been able to discuss \(\tau_m(g)\) on a fixed space \(M^N_n\) and \(\Gamma_n\). To avoid this inconvenience let

\[\mathcal{M} = \bigcap_{n, N \geq 1} M^N_n, \quad \Gamma = \bigcup_{n \geq 1} \Gamma_n\]

and we say a sequence \(\{m_k\}\) of \(\mathcal{M}\) converges to \(m\) in \(\mathcal{M}\) if \(m_k,e\) and \(m_k,o\) converge in \(M^N_n\) as \(k \to \infty\) for every \(n, N \geq 1\). Then, what we have proved is

1. \(\Gamma\) act on \(\mathcal{M}\) continuously.
2. \(\tau_m(g) > 0\) for any \(m \in \mathcal{M}\) and \(g \in \Gamma\).
3. If \(g_t = e^{th} \in \Gamma\), then \(\tau_m(g_t)\) is infinitely differentiable.
4. \(q(t, x) = -2\partial_x^2 \log \tau_m(e^{tx-4tx^2})\) is \(C^\infty\) and satisfies the KdV equation.

and

\[Q = \{ q : q(x) = -2\partial_x^2 \log \tau_m(e^x) \quad \text{for } m \in \mathcal{M} \},\]

and define for \(q \in Q\) and \(g \in \Gamma\)

\[(K(g)q)(x) = -2\partial_x^2 \log \tau_m(g e^x).\]

**Theorem 39** \(\{K(g)\}_{g \in \Gamma}\) defines a flow on \(Q\). For any one parameter group \(g_t(z) = e^{th(z)} \in \Gamma\) the function \((K(g_t)q)(x)\) is \(C^\infty\)-differentiable as a function of two variables. This \(\{K(g)\}_{g \in \Gamma}\) satisfies

\[\begin{cases}
(K(e^{t^2}q))(x) = q(x + t) \\
(K(e^{-4tx^2})q)(x) = q(x, t) \quad \text{a solution to the KdV equation} \\
(K(e^{t^{2n+1}}q))(x) = q_n(x, t) \quad \text{a solution to the } n\text{-th KdV equation}
\end{cases}\]
The definition of $Q$ is indirect, so the next problem we have to give examples of $Q$. In Lemma 7 we have shown that if $q$ is in $C^N$ in a neighborhood of 0, then the Weyl functions $m_{\pm}$ have asymptotic expansions

\[
\begin{align*}
  m_+(z) &= -\sqrt{z} - \sum_{j=1}^{N+1} c_{j+1} (\sqrt{z})^{-j} + o \left( \sqrt{z}^{-N-1} \right) \\
  m_-(z) &= -\sqrt{z} - \sum_{j=1}^{N+1} \frac{(-1)^{j+1}}{\sqrt{z}} e_j (\sqrt{z})^{-j} + o \left( \sqrt{z}^{-N-1} \right)
\end{align*}
\]

(77)
as $z \to \infty$ in a sector $\{ \epsilon < \arg z < \pi - \epsilon \}$ with real numbers $c_j$. Since $m_e$, $m_o$, and $m_\pm$ are related with

\[
\begin{align*}
  m_e(z) &= \frac{1}{2} (m_+(z) - m_-(z)) \\
  m_o(z) &= -\frac{1}{2\sqrt{z}} (m_+(z) + m_-(z))
\end{align*}
\]
it is easy to see that $m \in M$ if and only if (77) holds on $C_{\omega}$ with $\omega$ such that $\omega(x) \sim (-x)^{-n/2+1}$ as $x \to -\infty$ for any $n, N \geq 1$. Therefore we have to find potentials satisfying this condition. If $q$ is fast decaying together with its derivatives, then its $m_{\pm}$ can be continuous even on $(0, \infty)$ and have expansions of (77) there. Therefore, certainly the Schwartz class $\mathcal{S} \subset Q$. Smooth periodic potentials also satisfy this condition. Since $m_{\pm}$ depend on $q$ on $\mathbb{R}$ respectively, for a smooth potential $q$ is in $Q$, if it is periodic on one of the half axes and rapidly decreasing on the opposite axis, which is called a step-like potential. In the next section we show that smooth ergodic potentials are elements of $Q$.

## 6 KdV equation with ergodic initial data

In this section we show smooth ergodic potentials are elements of $Q$, or equivalently they satisfy the asymptotics (77) on $C_{\omega}$. What we have in mind is the inequality in Lemma (21) which relates the reflection coefficient $R(z)$ with the Lyapunov exponent and the IDS. The following lemma suggests us the quantities which we have to consider.

**Lemma 40** For any complex numbers $m_\pm \in \mathbb{C}_+$ set

\[
m_1 = -\frac{1}{m_+ + m_-}, \quad m_2 = \frac{m_+ m_-}{m_+ + m_-}, \quad R = \frac{m_1 + m_-}{m_+ + m_-}.
\]

Then, $m_1, m_2 \in \mathbb{C}_+$, $|R| \leq 1$ hold, and $\xi_j = (\arg m_j) / \pi$ $(j = 1, 2)$ satisfy

\[
\left| \xi_1 - \frac{1}{2} \right|, \quad \left| \xi_2 - \frac{1}{2} \right| \leq \frac{2}{\pi} |R|.
\]

(78)

With this lemma in mind we set

\[
\begin{align*}
  m_1(z) &= -\frac{1}{m_+(z) + m_-(z)}, \quad m_2(z) = \frac{m_+(z)m_-(z)}{m_+(z) + m_-(z)} \\
  \xi_j(z) &= \frac{1}{\pi} \arg m_j(z) \left( = \frac{1}{\pi} \text{Im} \log m_j(z) \right), \quad (j = 1, 2), \text{ for } z \in \mathbb{C}_+.
\end{align*}
\]
Then, \( \{m_j\} \) are Herglotz functions, and the condition (77) can be replaced by

\[
\begin{aligned}
  m_1(-z) &= \frac{1}{2} \sqrt{z^{-1}} \left( 1 + \sum_{n=1}^{(N+1)/2} b_n z^{-n} \right) + o \left( z^{-\frac{(N+1)}{2}} \right) \\
  m_2(-z) &= -\frac{1}{2} \sqrt{z} \left( 1 + \sum_{n=1}^{(N+1)/2} c_n z^{-n} \right) + o \left( z^{-\frac{(N+1)}{2}} \right)
\end{aligned}
\]  

(79)

If \( \text{Im} \ z > 0 \), then \( \text{Im} \ m_j(z) > 0 \), hence \( 0 \leq \xi_j(z) \leq 1 \) holds. \( \log m_j \) are of Herglotz as well, since \( \text{Im} \log m_j(z) = \xi_j(z) \), hence

\[
\begin{aligned}
  m_1(z) &= \frac{1}{2} \sqrt{z^{-1}} \exp \left( \int_{\lambda_0}^{\infty} \frac{\xi_1(\lambda) - I_{\lambda>0}/2}{\lambda - z} d\lambda \right) \\
  m_2(z) &= -\frac{1}{2} \sqrt{z} \exp \left( \int_{\lambda_0}^{\infty} \frac{\xi_2(\lambda) - I_{\lambda>0}/2}{\lambda - z} d\lambda \right)
\end{aligned}
\]  

(80)

are valid. Here we have used the asymptotic behavior (79) held in a sector region. Therefore, to have (79) it is sufficient that

\[
\int_0^\infty \lambda^N \left| \xi_j(\lambda) - \frac{1}{2} \right| d\lambda < \infty, \quad (j = 1, 2)
\]

hold for every \( N \geq 1 \). This is due to the expansion

\[
- \int_{\lambda_0}^{\infty} \frac{f(\lambda)}{\lambda - z} d\lambda = \sum_{n=0}^{N-1} z^{-n-1} \int_{\lambda_0}^{\infty} \lambda^n f(\lambda) d\lambda + z^{-N} \int_{\lambda_0}^{\infty} \frac{\lambda^N f(\lambda)}{\lambda - z} d\lambda.
\]

Recall the definition of the reflection coefficient \( R(z) \)

\[
R(z) = \frac{m_+(z) + m_-(z)}{m_+(z) + m_-(z)}.
\]

Therefore, Lemma 40 implies that for (79) it is sufficient to have

\[
\int_0^\infty \lambda^N |R(\lambda)| d\lambda < \infty \quad \text{for any} \ N > 1.
\]  

(81)

The condition (81) requires estimates of \( R(z) \) on \( \mathbb{R} \), which is possible if \( q \) is in the Schwartz space \( \mathcal{S} \). If \( q \) is periodic, \( R(\lambda) \) vanishes on its spectrum \( \Sigma \), hence the condition (81) is satisfied if \( R \) vanishes on \( \Sigma_{ac} \), which is possible if \( q \) is smooth. More generally for ergodic potentials Proposition 20 of [35] shows that \( R(\lambda) \) vanishes on \( \Sigma_{ac} \), hence the condition (81) is satisfied if

\[
\int_{0, \infty}^{\infty} \lambda^N d\lambda < \infty
\]

holds, namely the spectral gaps has to be small, which is the case if \( q \) is smooth. However, this condition implies that the ac spectrum is large, and the condition restricts the possible potential class strongly. To relax this condition we consider the possibility to replace the condition on \( \mathbb{R} \) for \( R(\lambda) \) by a condition for \( R(z) \) on the curve \( C \).
Proposition 41 Assume
\[ \int_C |z^N R(-z)| \, |dz| < \infty \] (82)
holds for any \( N \geq 1 \) and for any integer \( n \geq 1 \) with a curve \( C = C_\omega \) satisfying \( \omega (x) \sim (-x)^{-n+1/2} \). Then, the \( m \) is an element of \( \mathcal{M} \).

Proof. Let \( \phi (z) \) be a conformal map \( \phi_k \) in Lemma 52 with \( k = n \). Without loss of generality we can assume \( a_k^2 > -\lambda_0 \). Then, the assumption (82) and Lemma 40 imply
\[
\int_{-\infty}^{0} |\lambda|^N \left| \xi_j (-\phi (\lambda)) - \frac{1}{2} \right| d\lambda < \infty \quad \text{for any } N \geq 1.
\] (83)
Since \( m_j (-\phi (-z)) \) \( (j = 1, 2) \) are Herglotz functions (on \( \mathbb{C} \)) and its argument on \( \mathbb{R} \) is \( \pi \xi_j (-\phi (-\lambda)) \), applying the formula (80) to \( m_j (-\phi (-z)) \) yields
\[
\begin{align*}
m_1(-\phi (-z)) &= -\frac{1}{2\sqrt{-z}} \exp \left( \int_{-a_1^2}^{\infty} \frac{\xi_1 (-\phi (-\lambda)) - I_{\lambda>0}/2}{\lambda - z} d\lambda \right), \\
m_2(-\phi (-z)) &= -\frac{\sqrt{-z}}{2} \exp \left( \int_{-a_2^2}^{\infty} \frac{\xi_2 (-\phi (-\lambda)) - I_{\lambda>0}/2}{\lambda - z} d\lambda \right),
\end{align*}
\]
which shows that \( m_j (-\phi (-z)) \) satisfy the asymptotics (79) on \( C' \). Then, (102) of Lemma 52 implies that \( m_j (z) \) themselves satisfy (79) on \( C'' \) any curve similar to \( C' \), which completes the proof. \( \blacksquare \)

Remark 42 \( \xi_1 (\lambda) \) was used effectively by [12] for the study of inverse spectral problems, and they called it the xi function.

To apply Proposition 41 to ergodic potentials we need a lemma.

Lemma 43 Suppose the Lyapunov exponent \( \gamma (\lambda) \) satisfies
\[ \int_{0}^{\infty} \lambda^N \gamma (\lambda) \, d\lambda < \infty, \quad \text{for any } N \geq 1. \] (84)
Then, for a.e. \( \omega \in \Omega \) the condition (82) is fulfilled by \( R(z, \omega) \).

Proof. Recall (37):
\[
w(z) = -\sqrt{-z} = -\frac{c}{\sqrt{-z}} + \frac{1}{\sqrt{-z}} \int_{\lambda_0}^{0} \sqrt{-\lambda N (\lambda)} \, d\lambda + \frac{1}{\pi \sqrt{-z}} \int_{0}^{\infty} \frac{\sqrt{\lambda} \gamma (\lambda)}{\lambda - z} \, d\lambda
\]
with \( c = E(q_\omega(0)/2) \) (due to \( c_2 = q(0)/2 \)). Set
\[
\rho (\lambda) = \sqrt{-\lambda N (\lambda) I_{(\lambda_0,0]} (\lambda) + \frac{1}{\pi} \sqrt{-\lambda} \gamma (\lambda) I_{(0,\infty)} (\lambda)).
\]
Then, \( w \) is
\[
w(z) = -\sqrt{-z} - \frac{c}{\sqrt{-z}} + \frac{1}{\sqrt{-z}} \int_{\lambda_0}^{\infty} \frac{\rho (\lambda)}{\lambda - z} \, d\lambda,
\]
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and for any $m \geq 1$

$$w(z) = -\sqrt{-z} - \frac{c}{\sqrt{-z}} - \frac{1}{\sqrt{-z}} \sum_{k=0}^{m-1} z^{-k-1} \int_{\lambda_0}^{\infty} \lambda^k \rho(\lambda) d\lambda + \frac{1}{\sqrt{-z}} z^{-m} \int_{\lambda_0}^{\infty} \frac{\lambda^m \rho(\lambda)}{\lambda - z} d\lambda$$

(85)

holds due to the assumption (84). Recall the definition of $\chi$:

$$\chi(x + iy) = \frac{-\text{Re} w(x + iy)}{y} - \text{Im} w'(x + iy).$$

Taking (85) account into consideration, as a component of $\chi$ corresponding to $(-z)^{-k-1/2}$ we set

$$\chi_k(x + iy) = \frac{-\text{Re} (-x - iy)^{-k-1/2}}{y} + \left(k + \frac{1}{2}\right) \text{Im} (-x - iy)^{-k-3/2},$$

for $k \geq -1$. Noting

$$(-x - iy)^{-k-1/2} = i x^{-k-1/2} (-1)^k (1 + iyx^{-1})^{-k-1/2} = i x^{-k-1/2} (-1)^k \left(1 - (k + 1/2) iyx^{-1} + O(\text{yx}^{-1})^2\right)$$

for $x \geq 1$, $0 < y < 1$, we have

$$\chi_k(x + iy) = O(\text{yx}^{-k-5/2}).$$

(86)

On the other hand, the estimates

$$\left|\int_{\lambda_0}^{\infty} \frac{\lambda^m \rho(\lambda)}{(\lambda - z)^j} d\lambda\right| \leq y^{-j} \int_{\lambda_0}^{\infty} |\lambda^m \rho(\lambda)| d\lambda \quad (j = 1, 2)$$

yield a bound for the last term $w_x$ of (85). Let

$$\chi_x(x + iy) = \frac{-\text{Re} w_x(x + iy)}{y} - \text{Im} w'_x(x + iy).$$

Then, we have

$$\chi_x(x + iy) = O\left(y^{-2} x^{-m-1/2}\right).$$

(87)

(86), (87) imply

$$\chi(x + iy) = O\left(y^{-3/2} + y^{-2} x^{-m-1/2}\right).$$

This together with $\text{Im} w(x + iy) = O(\text{x}^{1/2})$ (due to $N(\lambda) \sim \sqrt{\lambda}$ as $\lambda \to \infty$) yields

$$\sqrt{2} \chi(z) \text{Im} w(z) = O\left(y x^{-1} + y^{-2} x^{-m}\right)^{1/2} = O\left(y^{1/2} x^{-1/2} + y^{-1} x^{-m/2}\right).$$

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Therefore, if the curve is parametrized as \( x + i (-x)^{-\alpha} \) near \( x = -\infty \), we have
\[
\mathbb{E} \left( \int_C |z|^{N'} |R(z, \omega)| \, dz \right) \leq c \int_1^{\infty} x^{N'} \left( x^{-\alpha/2} x^{-1/2} + x^{\alpha - m/2} \right) \, dx < \infty,
\]
if
\[
N' - \alpha/2 - 1/2, \quad N' + \alpha - m/2 < -1
\]
hold, which is valid for sufficiently large \( \alpha \) and \( m \). Then, Fubini’s theorem implies the condition (82).

**Theorem 44** Let \( \{q_\omega(x) = q(\theta_x \omega)\} \) be an ergodic process on \((\Omega, F, P, \{\theta_x\}_{x \in \mathbb{R}})\). Suppose the Lyapunov exponent \( \gamma(\lambda) \) satisfies
\[
\int_0^\infty \lambda^N \gamma(\lambda) \, d\lambda < \infty, \quad \text{for any } N \geq 1. \tag{88}
\]
Then, \( q_\omega \in Q \) for a.e. \( \omega \in \Omega \) holds. In this case \((K(g)q_\omega)(x) = f_g(\theta_x \omega)\) is valid with \( f_g(\omega) = (K(g)q_\omega)(0) \). Moreover, the condition (88) holds if \( q_\omega(x) \) is in \( C^\infty_b \).

**Proof.** From \( q_{\theta_x \omega}(y) = q_\omega(x+y) = (K(e_x)q_\omega)(y) \)
\[
f_g(\theta_x \omega) = (K(g)q_{\theta_x \omega})(0) = (K(g)K(e_x)q_\omega)(0) = (K(e_x)K(g)q_\omega)(0) = K(g)q_\omega(x)
\]
follows. Under the condition \( q_\omega \in C^\infty_b \), Lemma 22 implies (88).

In this way we can construct the KdV-flow on smooth ergodic processes.

**Remark 45** This theorem implies that \( C^\infty \)-smoothness follows from the condition (88). And if \( q_\omega \) is almost periodic, or equivalently \( \Omega \) is an infinite dimensional torus, then \((K(g)q_\omega)(x)\) is also almost periodic for any \( g \in \Gamma \).

### 7 Some questions

In this section we explain several questions which the author has in mind.

#### 7.1 Transfer matrices and reflectionless condition

We have defined a \( \Gamma \)-action \( g \cdot m \) on \( \mathcal{M} \) in Proposition 34. This action can be described also by transfer matrices, that is, there exists matrices
\[
\{T_m(z,g)\}_{m \in \mathcal{M}, g \in \Gamma} \subset SL(2, \mathbb{C})
\]
satisfying
\[
\begin{align*}
(1) \quad & T_m(z,g) \text{ is entire as a function of } z. \\
(2) \quad & (g \cdot m)(z) = T_m(z,g) \cdot m(z). \\
(3) \quad & T_m(z,g_1g_2) = T_{g_1 \cdot m}(z,g_2) T_m(z,g_1). \\
(4) \quad & T_m(x,g) \in SL(2, \mathbb{R}) \quad \text{if } x \in \mathbb{R}.
\end{align*}
\]
As a corollary of the existence of \( T_m(z,g) \), the **reflectionless property** is **preserved** under this action, namely for an set \( A \in \mathcal{B}(\mathbb{R}_+) \) with \( |A| > 0 \) if \( m_{\pm} \) are reflectionless, which is equivalent to
\[
m(0+iy) = m(-0+iy) \quad \text{for a.e. } y \in \sqrt{A},
\]

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then

$$(g \cdot m) (0 + iy) = (g \cdot m) (-0 + iy)$$

for a.e. $y \in \sqrt{A}$

holds, which means the reflectionless property of $g \cdot m$. This observation leads us to the other theorem guarantees the condition of $\mathcal{M}$. For $m = (m_{\pm})$ let

$$\Sigma_{refl} (m) = \left\{ \lambda \in \mathbb{R}; m_+ (\lambda + i0) = -m_- (\lambda + i0) \right\}.$$ 

Since $\text{Re} m_1 (\lambda + i0) = \text{Re} m_2 (\lambda + i0) = 0$ for a.e. $\lambda \in \Sigma_{refl} (m)$ hold, we have

$$\xi_1 (\lambda) = \xi_2 (\lambda) = \frac{1}{2}$$

for a.e. $\lambda \in \Sigma_{refl} (m)$,

hence, if we assume

$$\int_{[0,1]} \left| \Sigma_{ac} \right| d\lambda < \infty$$

for any $N \geq 1$, (89)

then

$$\int_0^\infty \left| \xi_j (\lambda) \right| - \frac{1}{2} d\lambda = \int_{[0,1]} \left| \xi_j (\lambda) \right| - \frac{1}{2} d\lambda \leq \frac{1}{2} \int_{[0,1]} \left| \Sigma_{ac} \right| d\lambda < \infty$$

holds for any $N \geq 1$. Therefore, (89) is a sufficient condition for $m$ to be an element of $\mathcal{M}$. The condition (89) is interesting due to the trace formula

$$q (0) = 2 \int_{\lambda_0}^{\infty} (I_{\lambda >0}/2 - \xi_1 (\lambda)) d\lambda,$$

which can be shown from (80) and the asymptotics of $m_{\pm}$. Therefore, a bound

$$\left| q (0) \right| \leq |\lambda_0| + ||[0, \infty) \setminus \Sigma_{refl} (m)||$$

is valid. Since $\Sigma_{refl} (m)$ is invariant under $\Gamma$-action or equivalently the KdV flow $K (g)$, one has

**Theorem 46** Assume a potential $q$ has its $\Sigma_{refl} (m)$ satisfying (89). Then, $q \in \mathcal{Q}$ and a uniform bound

$$|\left( K (g) q \right) (x) | \leq |\lambda_0| + ||[0, \infty) \setminus \Sigma_{refl} (m)||$$

is valid for any $g \in \Gamma$.

A version of the above theorem for ergodic potentials is

**Theorem 47** Suppose $\Omega$ is a metric space and the support of the ergodic probability measure $\mu$ coincides with $\Omega$. For a bounded continuous function $Q (\omega)$ on $\Omega$, assume the ergodic potential $q_\omega (x) = Q (T_x \omega)$ has ac spectrum $\Sigma_{ac}$ satisfying

$$\int_{[0,1]} \left| \Sigma_{ac} \right| d\lambda < \infty$$

for any $N \geq 1$. (91)

Then, $q_\omega \in \mathcal{Q}$ holds for any $\omega \in \Omega$. Moreover, $q_\omega$ satisfies

$$\left| \left( K (g) q_\omega \right) (x) \right| \leq |\lambda_0| + ||[0, \infty) \setminus \Sigma_{ac}||$$

for any $\omega \in \Omega$ and $g \in \Gamma$. 

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Proof. Since the reflectionless property
\[ m^+_m (\lambda + i0) = -m^+_m (\lambda + i0) \quad \text{for a.e. } \lambda \in \Sigma_{ac} \] (92)
holds for a.e. \( \omega \in \Omega \), one can know that (92) is valid for any \( \omega \in \Omega \) by approximating this \( \omega \) by \( \omega_s \) satisfying (92), hence \( q_{\omega} \in Q \) holds for any \( \omega \in \Omega \).

Remark 48 It might be verified that \((K(g)q_{\omega}) (0)\) in Theorem 44 is smooth if \( \Omega \) is a manifold and \( Q \) is smooth.

Another interesting problem is a generalization of Remling theorem established for the first KdV equation whose solution is given by the shift. In the proof the existence of transfer matrices was essential, hence it is natural to ask if this theorem holds for the KdV flow. However, another crucial property for the proof is the property \( T_m (z; g) : C_+ \rightarrow C_+ \) as a fractional linear transformation, which is valid for the shift (see Lemma 6), and it is known that this property holds in a restricted sense for the KdV flow. This result will be explained in another opportunity.

7.2 Almost periodicity in time

Deift [7] gave a conjecture that for any almost periodic initial data the solution to the KdV equation if it exists is almost periodic in time. This conjecture was verified by [33] and [34] in some cases. Their assumption includes the pure ac property of the spectrum, hence the reflectionless property holds on the spectrum. However, if some singular part is contained in the spectrum, the situation seems to be quite different. There is some possibility that this conjecture holds if the condition (91) is valid.

As was pointed out by Lax, if we consider two Schrödinger operators \( L_{q_s} \) and \( L_{q_t} \) with a solution \( q_t (x) \) to the KdV equation, then \( L_{q_s} \) and \( L_{q_t} \) are unitarily equivalent, hence the spectral structure of these two operators is exactly the same. Therefore, it would be interesting to investigate the role played by the spectrum in the study of asymptotic analysis of solutions to the KdV equation when the time \( t \) tends to \( \infty \). Remling theorem and Deift conjecture can be regarded as two significant problems related to this point of view.

In this respect one can ask the following question. Let \( \Pi \) be the set of all shift-invariant probability measures on \( Q \) which are ergodic and \( \{g_t\}_{t \in \mathbb{R}} \) be a one-parameter group on \( \Gamma \). For \( g \in \Gamma \) and \( \mu \in \Pi \) let \( K(g)^* \mu \) is the pushed forward measure of \( \mu \) defined by
\[ (K(g)^* \mu) (A) = \mu (K(g)^{-1} (A)) \quad \text{for } B (Q) . \]

Problem 49 For a given \( \mu \in \Pi \) find the limit set
\[ \mathcal{L}_\mu = \left\{ \tilde{\mu}; \lim_{n \to \infty} \frac{1}{t_n} \int_{0}^{t_n} K (g_s)^* \mu ds \text{ for some } t_n \to \infty \right\} . \]
\( \tilde{\mu} \) is necessarily invariant with respect to \( K (g_t) \) if \( \tilde{\mu} \in \mathcal{L}_\mu \). To investigate \( \mathcal{L}_\mu \) it would be interesting to consider the Floquet exponent. For each \( \mu \in \Pi \) one can define the Floquet exponent by
\[ w_\mu (z) = E_\mu (m_+ (z, q)) = \int_{Q} m_+ (z, q) \mu (dq) . \]
Since Theorem 44 implies that the pushed forward measure $K(g)^*\mu$ is also shift invariant and ergodic, one can define $w_{K(g)^*\mu}$. Then, the proposition below says that $w_{\mu}$ works as an invariant of $K(g)$, namely we have

**Proposition 50** $w_{\mu}(z) = w_{K(g)^*\mu}(z)$ holds for any $g \in \Gamma$ and $\mu \in \Pi$.

**Proof.** We show the statement only when $\text{supp} \mu \subset M_r^{\text{refl}}$ for some $r > 0$. Set
\[
 f(x, z, q) = e^{-xz} \tau_{e_x} w_{m}(g_z) \quad \text{for } m \in M_r^{\text{refl}}.
\]
Then, Lemma 32, Lemma 37 of [35] show
\[
 m(z, \theta_x q) = - \frac{\partial_x f(x, z, q)}{f(x, z, q)},
\]
where $\theta_x$ is the shift operation for potentials $((\theta_x q)(\cdot) = q(\cdot + x))$. Since $f(x, z, q)$ satisfies
\[
 -\partial_x^2 f(x, z, q) + q(x)f(x, z, q) = -z^2 f(x, z, q),
\]
the Ricatti equation
\[
 \partial_x m(z, \theta_x q) = -z^2 - q(x) + m(z, \theta_x q)^2 \quad (93)
\]
holds. On the other hand, if $|\zeta| > r$ is satisfied, then Lemma 55 of [35] implies
\[
 m(z, K(q_\zeta)) = (d\zeta_m)(z, q) = \frac{z^2 - \zeta^2}{m(z, q) - m(\zeta, q)} + m(\zeta, q). \quad (94)
\]
Assume $\zeta \in \mathbb{R}$ and $|\zeta| > r$. Then, $m(\zeta, \theta_x q) \in \mathbb{R}$ and $m(z, \theta_x q) - m(\zeta, \theta_x q) \neq 0$ if $z \in \mathbb{C}_+$, hence, from (93), (94) we have
\[
 \partial_x \log (m(z, \theta_x q) - m(\zeta, \theta_x q)) = m(z, \theta_x q) - \frac{z^2 - \zeta^2}{m(z, \theta_x q) - m(\zeta, \theta_x q)} + m(z, \theta_x q) = m(z, \theta_x q) - m(z, K(q_\zeta) \theta_x q).
\]
Taking the expectation yields
\[
w_{\mu}(-z^2) = \mathbb{E}_\mu (m(z, K(q_\zeta) g)) = w_{K(q_\zeta)^*\mu}(-z^2) \quad \text{if } \text{Re } z < 0.
\]
Here we have used the fact that $m(z, K(q_\zeta) q) = m_{K, W_m}(z) \in M_r^{\text{refl}}$ holds if $\zeta \in \mathbb{R}$ and $|\zeta| > r$. Repeating this procedure we have
\[
w_{\mu}(-z^2) = \mathbb{E}_\mu (m(z, K(q_{\zeta_1}, q_{\zeta_2}) q)) \quad (95)
\]
for $\zeta_1, \zeta_2 \in \mathbb{R}$ satisfying $|\zeta_1|, |\zeta_2| > r$. Since the right hand side of (95) has meaning $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfying $|\zeta_1|, |\zeta_2| > r$ and analytic there, it holds that
\[
w_{\mu}(-z^2) = \mathbb{E}_\mu \left(m \left( z, K \left(q_{\zeta_1}, q_{\zeta_2} \right) q \right) \right) = w_{K(q_{\zeta_1}, q_{\zeta_2})}(-z^2),
\]
where note $m \left( z, K \left(q_{\zeta_1}, q_{\zeta_2} \right) q \right) \in M_r^{\text{refl}}$, which leads us to the identity
\[
w_{\mu}(-z^2) = w_{K(g)^*\mu}(-z^2)
\]
for any $g \in \Gamma$ by induction and taking limit. ■
8 Appendix

8.1 Herglotz functions

A holomorphic function \( m \) on \( \mathbb{C}^+ \) is called a Herglotz function if it satisfies 

\[
\text{Im} \, m(z) > 0 \quad \text{for any } z \in \mathbb{C}^+.
\]

If \( m \) is a Herglotz function, \( \text{Im} \, m(z) \) is a positive harmonic function on \( \mathbb{C}^+ \), hence it is known that there exists a measure \( \sigma \) on \( \mathbb{R} \) and \( \beta \geq 0 \) such that

\[
\text{Im} \, m(z) = \beta y + \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} \sigma(d\lambda)
\]

with \( z = x + iy \). The measure \( \sigma \) should satisfy

\[
\int_{-\infty}^{\infty} \frac{1}{1 + \lambda^2} \sigma(d\lambda) < \infty.
\]

Since an identity

\[
\beta y + \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} \sigma(d\lambda) = \text{Im} \left( \beta z + \int_{-\infty}^{\infty} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) \sigma(d\lambda) \right)
\]

holds, there exists a real \( \alpha \) such that

\[
m(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) \sigma(d\lambda).
\]

This is the representation theorem of a Herglotz function. The measure \( \sigma \) is called a spectral measure of \( m \). The following facts are known for Herglotz functions.

**Lemma 51**

1. \( \lim_{t \to 0} \int_a^b \text{Im} \, m(\lambda + ie) \, d\lambda = \pi \sigma([a,b]) \) if \( \sigma([a]) = \sigma([b]) = 0 \).
2. \( \lim_{t \to 0} \int_{-\infty}^{\infty} \text{Im} \, m(\lambda + ie) = \pi \sigma'(\lambda) \) for a.e. \( \lambda \in \mathbb{R} \), where \( \sigma'(\lambda) \, d\lambda \) is the absolutely continuous part of \( \sigma \).
3. \( m(\lambda + 0) = \lim_{t \to 0} m(\lambda + ie) \) exists finitely for a.e. \( \lambda \in \mathbb{R} \) and satisfies \( m(\lambda + 0) \neq 0 \). More generally if two Herglotz functions \( m_1, m_2 \) satisfy \( m_1(\lambda + 0) = m_2(\lambda + 0) \) on some \( A \) with positive Lebesgue measure, then it holds that \( m_1 = m_2 \) identically.

8.2 Conformal maps from \( \mathbb{C}\setminus(-\infty,0] \) to \( D_- \)

Although Riemann mapping theorem says that every simply connected domain on \( \mathbb{C} \) can be an image of a conformal map on \( \mathbb{C}^+ \), sometimes a quantitative estimate of it is necessary. In this section we provide a model of conformal map from \( \mathbb{C}\setminus(-\infty,0] \) to \( D_- \) of (45).

A conformal map \( \psi \) on \( \mathbb{C}^+ \) is easily obtained if \( \text{Im} \, \psi'(z) \) has a definite sign on \( \mathbb{C}^+ \). A simple such example is \( \psi_\infty(z) = \sqrt{z} \), and a more general conformal map in this framework can be constructed for an integer \( k \geq 1 \) by an integral

\[
\psi_k(z) = \frac{k - 1/2}{k + 1/2} \sqrt{z} + \int_0^\infty \sqrt{z + t}(1 + t)^{-k-3/2} \, dt.
\]
This \( \psi_k \) satisfies
\[
\text{Re} \, \psi_k(z) > 0, \quad \text{Im} \, \psi_k(z) > 0, \quad \text{Re} \, \psi_k'(z) > 0, \quad \text{Im} \, \psi_k'(z) < 0
\] (97)
for \( z \in \mathbb{C}_+ \), hence \( \psi_k \) maps \( \mathbb{C}_+ \) to \( \psi_k(\mathbb{C}_+) \subset \mathbb{C}_+ \), and \( \phi_k(z) = \psi_k(z)^2 \) maps \( \mathbb{C}_+ \) to \( \phi_k(\mathbb{C}_+) \subset \mathbb{C}_+ \) conformally. Since \( \psi_k(z) \) takes real values on \([0, \infty)\), \( \psi_k(z) \) and \( \phi_k(z) \) can be extended as conformal maps from \( \mathbb{C}\setminus(-\infty,0) \) onto a domain in \( \{\text{Re} \, z > 0\} \) and a domain in \( \mathbb{C} \) respectively. Set
\[
\begin{align*}
  a_k &= 2 \int_0^1 s^2 (1 - s^2)^{k-1} \, ds = \frac{\sqrt{\pi} \Gamma(k)}{2 \Gamma(k + 3/2)} \\
  b_k &= 2a_k \left( 2a_k k (k + 1/2)^2 / (k - 1/2)^2 + 1 \right)^{-1/2}
\end{align*}
\]
The image \( \phi_k(\mathbb{C}\setminus(-\infty,0)) \) is a type of \( D_\omega \) in 5.2.1.

**Lemma 52** The image \( \phi_k(\mathbb{C}\setminus(-\infty,0]) \) is described as follows:
\[
\phi_k(\mathbb{C}\setminus(-\infty,0]) = \mathbb{C} \setminus \{ z \in \mathbb{C}; \ |\text{Im} \, z| \leq \omega(\text{Re} \, z), \ \text{Re} \, z \leq a_k^2 \}
\] (98)
with positive smooth function \( \omega(x) \) on \((-\infty, a_k^2)\) such that
\[
\omega(x) = \begin{cases} 
  2a_k (-x)^{-k+1/2} \left( 1 + O(x^{-1}) \right) & \text{as } x \to -\infty \\
  b_k (a_k^2 - x)^{1/2} \left( 1 + O(a_k^2 - x) \right) & \text{as } x \to a_k^2 - 0 
\end{cases}
\] (99)
Moreover, \( \phi_k \) takes a form of
\[
\phi_k(z) = z + f_1(z) + z^{-k+1/2} f_2(z)
\] (100)
with some real rational functions \( f_1, f_2 \) (that is, \( f_j(z) = \overline{f_j(\overline{z})} \) for \( j = 1, 2 \)) satisfying
\[
\begin{align*}
  f_1(\infty) &= (k^2 - 1/4)^{-1} \\
  f_2(\infty) &= 2 (-1)^k a_k
\end{align*}
\] (101)
Conversely, \( \phi_k^{-1}(w) \) has an expression
\[
\phi_k^{-1}(w) = w + g_1(w) + w^{-k+1/2} g_2(w)
\] (102)
with \( g_1, g_2 \) holomorphic in a neighborhood of \( \infty \). Moreover, it holds that
\[
g_1(\infty) = - (k^2 - 1/4)^{-1}, \quad g_2(\infty) = -2 (-1)^k a_k.
\] (103)

**Proof.** Setting \( s = \sqrt{(z + t)/(1 + t)} \), we have
\[
\psi_k(z) = \frac{k - 1/2}{k + 1/2} \sqrt{z + 2(z - 1)^{-k}} \int_1^{\sqrt{z}} s^2 (s^2 - 1)^{k-1} \, ds.
\]
Since the integral \( \int_0^z s^2 (s^2 - 1)^{k-1} \, ds \) is an odd polynomial of degree \( 2k + 1 \), the integral
\[
p(z) = \frac{k - 1/2}{k + 1/2} (z - 1)^{k+1} 2 \sqrt{z}^{-1} \int_0^{\sqrt{z}} s^2 (s^2 - 1)^{k-1} \, ds
\] (104)
defines a polynomial of degree \( k \), and
\[
\psi_k (z) = (z - 1)^{-k} (\sqrt{z} p (z) - p (1)) \tag{105}
\]
holds. It should be noted that \( \sqrt{z} p (z) - p (1) \) has zero of degree \( k \) at \( z = 1 \), so \( \psi_k (z) \) has no singularity at \( z = 1 \). Set
\[
s(x) = \Re \psi_k (x + i 0), \quad t(x) = \Im \psi_k (x + i 0)
\]
for \( x \in \mathbb{R} \). Then, (105) implies
\[
s(x) = \begin{cases} 
-p(1) (x - 1)^{-k} & \text{for } x < 0 \\
(x - 1)^{-k} (\sqrt{x} p(x) - p (1)) & \text{for } x \geq 0
\end{cases},
\]
\[
t(x) = \begin{cases} 
(x - 1)^{-k} \sqrt{x} p(x) & \text{for } x < 0 \\
0 & \text{for } x \geq 0
\end{cases},
\]
and their asymptotics are
\[
s(x) = \begin{cases} 
\frac{a_k (1 + kx + O (x^2))}{x} & \text{as } x \to -0 \\
\frac{a_k (-x)^{-k} (1 + kx^{-1} + O (x^{-2}))}{x} & \text{as } x \to -\infty
\end{cases},
\]
\[
t(x) = \begin{cases} 
\frac{k - 1/2}{k + 1/2} \sqrt{x} (1 + O (x)) & \text{as } x \to -0 \\
\sqrt{-x} (1 + O (x^{-1})) & \text{as } x \to -\infty
\end{cases}, \tag{106}
\]
where we have used
\[
\begin{cases} 
p(1) = (-1)^{k-1} a_k, \\
p(0) = \frac{k - 1/2}{k + 1/2} (-1)^k \\
p(z) = z^k - \left( k - \frac{2}{4k^2 - 1} \right) z^{k-1} + \ldots
\end{cases}, \tag{107}
\]
From (105)
\[
\phi_k (z) = \psi_k (z)^2 = z + f_1 (z) + z^{-k+1/2} f_2 (z)
\]
follows, which yields (100) with
\[
\begin{cases} 
f_1(z) = (z - 1)^{-2k} \left( p(1)^2 + z p(z)^2 \right) - z \\
f_2(z) = -2 (z - 1)^{-2k} p(1) p(z) z^k
\end{cases},
\]
and
\[
\begin{cases} 
\Re \phi_k (x + i 0) = x + f_1 (x) + f_2 (x) \times \begin{cases} x^{-k+1/2} & \text{if } x > 0 \\
0 & \text{if } x < 0
\end{cases} \\
\Im \phi_k (x + i 0) = \begin{cases} 0 & \text{if } x > 0 \\
\sqrt{-x} x^{-k} f_2 (x) & \text{if } x < 0
\end{cases}
\end{cases} \tag{108}
\]
is valid, hence (107) shows
\[
\begin{cases} 
\Re \phi_k (x + i 0) = x + \left( k^2 - 1/4 \right)^{-1} + O \left( (-x)^{-1} \right) \\
\Im \phi_k (x + i 0) = 2a_k (-x)^{-k+1/2} + O \left( (-x)^{-k-1/2} \right)
\end{cases} \tag{109}
\]
as \( x \to -\infty \), and

\[
\begin{align*}
\text{Re} \phi_k(x + i0) &= a_k^2 + \left(2ka_k^2 + \left(\frac{k-1/2}{k+1/2}\right)^2\right)x + O(x^2) \\
\text{Im} \phi_k(x + i0) &= \frac{2k-1/2}{k+1/2}a_k\sqrt{-x} + O\left((-x)^{3/2}\right)
\end{align*}
\]

(110)
as \( x \to -0 \). Since \( \text{Re} \phi_k(x + i0) = s(x)^2 - t(x)^2 \), \( \text{Im} \phi_k(x + i0) = 2s(x)t(x) \), (97) implies

\[
\begin{align*}
\text{Re} \phi_k(x + i0) &\text{ is increasing and moving from } -\infty \text{ to } \infty \\
\text{Im} \phi_k(x + i0) &> 0 \text{ on } (-\infty, 0) \text{ and } 0 \text{ on } [0, \infty)
\end{align*}
\]

(111)Therefore, \( \omega \) can be defined by an equation

\[
\omega(\text{Re} \phi_k(x + i0)) = \text{Im} \phi_k(x + i0).
\]
due to (111), and (108), (110), (111) show \( \omega(x) \) satisfies (99).

We use (105) to show (102). Set \( \vartheta(z) = z^2 \). \( \vartheta \) is a conformal map from \( \{ \text{Re } z > 0 \} \to \mathbb{C}\setminus(-\infty,0] \) and define \( \tilde{\psi}_k(s) = \psi_k(\vartheta(s)) \). Then the function

\[
F(s) = \frac{\tilde{\psi}_k(s)}{\vartheta(s)} - s
\]

= \(-p(1)(s^2 - 1)^{-k} + s\left[(s^2 - 1)^{-k} + s\right] - 1\),
is a rational function whose poles only at \( s = \pm 1 \) and has expansion

\[
F(s) = c_1s^{-1} + c_2s^{-3} + \cdots + c_{k}s^{-2k+1} + c_{k+1}s^{-2k+1}
\]
at \( s = \infty \) with \( c_1 = (2k^2 - 1/2)^{-1} \) and \( c_{k+1} = -p(1) \), namely the first coefficient of even order starts from \( 2k \). We consider an equation for a given \( t \):

\[
s + F(s) = t
\]

(112)and find a solution of a form

\[
s = t + G(t).
\]

Since the even coefficients of the power series of \( F \) vanish up to \( 2(k - 1) \), Lemma 53 shows that there exists uniquely such \( G \) that \( G \) is holomorphic near \( t = \infty \) and the odd coefficients of \( G \) vanishes up to \( 2(k - 1) \). \( \psi_k(z) \) is one-to-one on \( \{ |z| > r_1 \} \) and its inverse is given by \( w + G(w) \) on \( \{ |w| > r_2 \} \). Since \( \phi_k(z) = \vartheta(\psi_k(z)) \) is a conformal map from \( \mathbb{C}\setminus(-\infty,0] \) to \( \phi_k(\mathbb{C}\setminus(-\infty,0]) \), its inverse is given by \( \phi_k^{-1}(w) = \vartheta^{-1}(\psi_k^{-1})(w) \) for \( w \in \phi_m(\mathbb{C}\setminus(-\infty,0]) \). Let

\[
G(t) = G_e(t^2) + tG_o(t^2).
\]

Then, we have

\[
\left(\vartheta^{-1}\psi^{-1}\right)(w)
\]

= \( \left(\sqrt{w + G_e(w)} + \sqrt{G_o(w)}\right)^2 \)

= \( w + G_e(w)^2 + w\left(G_o(w) + 1\right)^2 - 1\) + \( 2\sqrt{G_e(w)}(G_o(w) + 1) \).

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Since \( G \) has the vanishing even coefficients up to \( 2(k - 1) \), \( w^k G_e(w) \) is holomorphic near \( w = \infty \). Therefore, setting
\[
\begin{align*}
  g_1(w) &= G_e(w)^2 + w G_o(w) (G_o(w) + 2) \\
  g_2(w) &= 2 w^k G_e(w) (G_o(w) + 1)
\end{align*}
\]
we have
\[
\phi_k^{-1}(w) = w + g_1(w) + w^{-k+1/2} g_2(w)
\]
with some \( g_1, g_2 \) holomorphic in a neighborhood of \( \infty \) satisfying
\[
g_1(\infty) = - (k^2 - 1/4)^{-1}, \quad g_2(\infty) = -2 (-1)^k a_k,
\]
which completes the proof. \( \square \)

**Lemma 53** Let \( F \) be a power series of \( s^{-1} \) given by \( F(s) = \sum_{j=1}^{\infty} a_j s^{-j} \) and assume it has the positive radius of convergence and consider an equation:
\[
t = s + F(s).
\]

(i) This equation is uniquely solvable if \( |t^{-1}| \) is sufficiently small and it has a form:
\[
s = t + G(t)
\]
with a convergent power series of \( t^{-1} \) given by
\[
G(t) = \sum_{j=1}^{\infty} x_j t^{-j}.
\]

(ii) \( x_n \) is determined from \( \{a_j\}_{j=1}^{n} \) for each \( n \geq 1 \). The first three coefficients are
\[
x_1 = -a_1, \quad x_2 = -a_2, \quad x_3 = -a_1^2 - a_3.
\]

(iii) Suppose \( F(s) \) has a form
\[
F(s) = \sum_{j=1}^{k} a_{2j-1} s^{-2j+1} + \sum_{j=2k}^{\infty} a_j s^{-j}
\]
for an \( k \geq 1 \). Then, the coefficients \( x_j \) of \( G(t) \) vanish for even \( j \) up to \( 2(k - 1) \). Moreover, if \( a_{2j} \neq 0 \), then \( x_{2j} = -a_{2j} \).

**Proof.** Replacing \( s \) by \( s^{-1} \) and \( t \) by \( t^{-1} \) we see the equation (113) is equivalent to
\[
t = \frac{s}{1 + s F(s^{-1})}.
\]
The condition on \( F \) implies
\[
t(0) = 0, \quad \frac{dt}{ds}(0) = 1, \quad \frac{d^2 t}{ds^2}(0) = 0,
\]
hence the complex function theory shows the existence of the solution \( s(t) \) of (116) in a neighborhood of 0 satisfying
\[
s(0) = 0, \quad \frac{ds}{dt}(0) = 1, \quad \frac{d^2 s}{dt^2}(0) = 0,
\]
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which implies the existence of $G$ of the form of (114). One can show inductively that the coefficient $x_n$ is determined from \( \{a_j\}_{j=1}^n \). To show (iii) one can assume $a_j = 0$ for every $j \geq 2k$ owing to (ii). The relation between $F, G$ is rewritten as

$$F(s) + G(F(s) + s) = 0.$$ 

If we define $\hat{f}(s) = -f(-s)$, then the above equation turns to

$$\hat{F}(s) + \hat{G}(\hat{F}(s) + s) = 0.$$ 

Therefore, if $\hat{F}(s) = F(s)$ the uniqueness implies $\hat{G}(s) = G(s)$, which shows the first part of (iii). To show the second part we note that if

$$F(s) = \sum_{j=1}^{k-1} a_j s^{-j} + a_k s^{-k} \equiv F_1(s) + a_k s^{-k},$$ 

$$G(s) = \sum_{j=1}^{k-1} x_j s^{-j} + x_k s^{-k} + \sum_{j=k+1}^{\infty} x_j s^{-j} \equiv G_1(s) + x_k s^{-k} + G_2(s)$$

and with some $b_m$

$$F_1(s) + G_1(s + F_1(s)) = b_k s^{-k} + O(s^{-k-1}) \quad (117)$$

holds, which is verified by induction, then the identity

$$F_1(s) + a_k s^{-k} + G_1(s + F_1(s) + a_k s^{-k}) + x_k (s + F_1(s) + a_k s^{-k})^{-k} + G_2(s + F_1(s) + a_k s^{-k}) = 0$$

together with

$$G_1(s + F_1(s) + a_k s^{-k}) = G_1(s + F_1(s)) + O(s^{-k-2})$$

implies $x_k = -a_k - b_k$. Since, if $k$ is even and $a_{2j} = 0$ for any $j \leq k/2$, then (ii) implies $x_j = 0$, and hence $b_k = 0$. However, clearly $b_k$ is determined from \( \{a_j\}_{1 \leq j \leq k-1} \), hence $b_k = 0$ is valid if $a_{2j} = 0$ for any $j \leq k/2 - 1$ regardless of the value $a_k$. Consequently we have $x_k = -a_k$ if $k$ is even and $a_{2j} = 0$ for any $j \leq k/2 - 1$ holds. 

### 8.3 Approximation

For two Herglotz functions $m_{\pm}$ define

$$m_+(z) = -\frac{1}{2} (m_+(-z) - m_-(z))$$

$$m_-(z) = -\frac{1}{2\sqrt{z}} (m_+(-z) + m_-(z))$$

and $m(z) = m_+(z^2) + zm_-(z^2)$. For $a < 0 < b$ let $C^{a,b}$ be a simple closed smooth curve cutting $\mathbb{R}$ at $a$, $b$ (see f.4), and set

$$m_{\pm}^{a,b}(z) = \frac{1}{2\pi i} \int_{C^{a,b}} m_\pm(\lambda) \frac{d\lambda}{z - \lambda}$$

$$m_0^{a,b}(z) = 1 + \frac{\sqrt{1-a/z}}{2\pi i} \int_{C^{a,b}} \frac{m_0(\lambda) - 1}{(z - \lambda) \sqrt{1-a/\lambda}} d\lambda$$

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and
\[ \begin{align*}
    m_{+}^{a,b}(z) &= -m_{e}^{a,b}(-z) - \sqrt{z}m_{o}^{a,b}(-z) \\
    m_{-}^{a,b}(z) &= m_{e}^{a,b}(-z) - \sqrt{z}m_{o}^{a,b}(-z) .
\end{align*} \]

**Lemma 54** $m_{\pm}^{a,b}(z)$ are of Herglotz and reflectionless on $(-a, \infty)$ and the representing measures of $m_{\pm}^{a,b}(z)$ vanish on $(-\infty, -b)$.

**Proof.** Let $m_{\pm}$ be
\[ m_{\pm}(z) = \alpha_{\pm} + \beta_{\pm}z + \int_{-\infty}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sigma_{\pm}(d\xi) . \]

Since
\[ \begin{align*}
    &\frac{1}{2\pi i} \int_{C^{a,b}} \frac{1}{\lambda - \lambda} \left( \frac{1}{\xi + \lambda} - \frac{\xi}{\xi^2 + 1} \right) d\lambda = \frac{I_{(-b,-a)}(\xi)}{\xi + z} \\
    &\frac{1}{2\pi i} \int_{C^{a,b}} \frac{\alpha_{\pm} - \beta_{\pm}}{\lambda - \lambda} d\lambda = 0 \\
    &\frac{1}{2\pi i} \int_{C^{a,b}} \frac{\sqrt{\lambda}}{\sqrt{\lambda} - a} d\lambda = \frac{1}{2\pi i} \int_{C^{a,b}} \frac{1}{(z - \lambda) \sqrt{1 - a/\lambda}} d\lambda
\end{align*} \]

for $z \in C^{a,b}_{out}$ the unbounded connected domain separated by $C^{a,b}$, we have
\[ m_{e}^{a,b}(z) = -\frac{1}{2\pi i} \int_{C^{a,b}} \frac{m_{+}(-\lambda) - m_{-}(-\lambda)}{2(\lambda - z)} d\lambda = -\int_{-b}^{-a} \frac{\sigma_{+}(d\xi) - \sigma_{-}(d\xi)}{2(\xi + z)} . \]

On the other hand, since $\sqrt{\lambda - a}$ is analytic on $\{\text{Re}\, \lambda > a\}$, for $z \in C^{a,b}_{out}$ identities
\[ \begin{align*}
    &\frac{1}{2\pi i} \int_{C^{a,b}} \frac{1}{\lambda - \lambda} \left( \frac{1}{\xi + \lambda} - \frac{\xi}{\xi^2 + 1} \right) d\lambda = \frac{I_{(-b,-a)}(\xi)}{\xi + z} \\
    &\frac{1}{2\pi i} \int_{C^{a,b}} \frac{\alpha_{\pm} - \beta_{\pm}}{\lambda - \lambda} d\lambda = 0 \\
    &\frac{1}{2\pi i} \int_{C^{a,b}} \frac{\sqrt{\lambda}}{\sqrt{\lambda} - a} d\lambda = \frac{1}{2\pi i} \int_{C^{a,b}} \frac{1}{(z - \lambda) \sqrt{1 - a/\lambda}} d\lambda
\end{align*} \]

imply that
\[ m_{o}^{a,b}(z) = 1 + \sqrt{1 - a} \frac{1}{2\pi i} \int_{C^{a,b}} \frac{(m_{+}(-\lambda) + m_{-}(-\lambda))}{(z - \lambda) \sqrt{1 - a/\lambda}} d\lambda \\
= \sqrt{1 - a/z} - \frac{1}{2} \sqrt{1 - a/z} \int_{-b}^{-a} \frac{1}{\xi + z} \frac{\sigma_{+}(d\xi) + \sigma_{-}(d\xi)}{\sqrt{a - \xi}} . \]

Therefore, we have
\[ m_{\pm}^{a,b}(z) = -\sqrt{z - a} + \int_{-b}^{-a} \frac{\sqrt{z - a} \sigma_{+}(d\xi) + \sigma_{-}(d\xi)}{\sqrt{a - \xi}} \pm \int_{-b}^{-a} \frac{\sigma_{+}(d\xi) - \sigma_{-}(d\xi)}{2(\xi - z)} . \]

(118)
Set
\[
\begin{align*}
\rho_0 (\xi) &= \frac{1}{\pi} I_{\xi > -a} \sqrt{\xi + a} \int_{-a}^{\xi - \eta} \frac{1}{\xi - \eta} \frac{\sigma_+ (d\eta) + \sigma_- (d\eta)}{2\sqrt{\xi - \eta}} \\
\rho_1 (\xi) &= \frac{1}{\pi} I_{\xi > -a} \sqrt{\xi + a} \\
\alpha_0 &= -\text{Im} \sqrt{a + i}
\end{align*}
\]

For \( m_{\pm}^{a,b} \) to be of Herglotz it is sufficient to show
\[
m_{\pm}^{a,b} (z) - \alpha_0 = \int_{-b}^{-a} \frac{\sigma_+ (d\xi)}{\xi - z} + \int_{-a}^{\infty} \frac{\rho_0 (\xi) d\xi}{\xi - z} + \int_{-a}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \rho_1 (\xi) d\xi \tag{119}
\]

Since
\[
\int_{-a}^{\infty} \frac{\rho_0 (\xi) d\xi}{\xi - z} = \frac{1}{\pi} \int_{-a}^{\infty} \frac{\sqrt{\xi + a}}{\xi - z} d\xi \int_{-b}^{-a} \frac{1}{\xi - \eta} \frac{\sigma_+ (d\eta) + \sigma_- (d\eta)}{2\sqrt{-a - \eta}}
\]
\[
= \int_{-b}^{-a} \frac{\sigma_+ (d\eta) + \sigma_- (d\eta)}{2\sqrt{-a - \eta}} \frac{1}{\pi} \int_{-a}^{\infty} \frac{\sqrt{\xi + a}}{\xi - z} \frac{1}{\xi - \eta} d\xi
\]
\[
= \int_{-b}^{-a} \frac{\sigma_+ (d\eta) + \sigma_- (d\eta)}{2\sqrt{-a - \eta}} \frac{\sqrt{-a - \eta}}{\sqrt{-a - \eta} - \sqrt{-a - z}}
\]
\[
= \frac{1}{2} \int_{-b}^{-a} \left( 1 - \frac{\sqrt{-a - \eta}}{\sqrt{-a - \eta} - \sqrt{-a - z}} \right) \frac{\sigma_+ (d\xi) + \sigma_- (d\xi)}{\xi - z}
\]
and
\[
\int_{0}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \rho_1 (\xi) d\xi = \frac{1}{\pi} \int_{-a}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sqrt{\xi + a} d\xi
\]
\[
= -\sqrt{\xi - z} - \alpha
\]

hold, the right hand side of (119) turns to
\[
-\sqrt{-a - z} + \int_{-b}^{-a} \frac{\sigma_+ (d\xi)}{\xi - z} + \int_{-b}^{-a} \left( \frac{\sqrt{-a - \xi}}{\sqrt{-a - \xi} - 1} \right) \frac{\sigma_+ (d\xi) + \sigma_- (d\xi)}{2(\xi - z)}
\]
\[
= -\sqrt{-a - z} \pm \int_{-b}^{-a} \frac{\sigma_+ (d\xi) - \sigma_- (d\xi)}{2(\xi - z)} + \sqrt{-a - z} \int_{-b}^{-a} \frac{\sigma_+ (d\xi) + \sigma_- (d\xi)}{2\sqrt{-a - \xi}(\xi - z)},
\]
which is equal to \( m_{\pm}^{a,b} (z) \). Due to the fact \( \rho_0 (\xi), \rho_1 (\xi) \geq 0 \) we see that \( m_{\pm}^{a,b} \) are Herglotz functions. The reflectionless property on \((-a, \infty)\) and the vanishing of the representing measures of \( m_{\pm}^{a,b} \) on \((-\infty, -b)\) follow from (118) immediately by applying
\[
\sqrt{-(\lambda + i0) - a} = \begin{cases} 
-\frac{i\sqrt{\lambda + a}}{\sqrt{-a - \lambda}} & \text{if } \lambda > -a \\
\frac{\sqrt{-a - \lambda}}{\sqrt{-a - \lambda}} & \text{if } \lambda < -b
\end{cases}
\]

We prepare several estimates of integrals which appear later. In the sequel \( c_j \) denotes some positive constant.
Lemma 55 Let $\epsilon > 0$ and $p, q > 0$ satisfy $p + q = 1$. For $y \geq 0$ set

\[ \begin{align*}
\varphi_1(y) &= \int_0^1 \frac{1}{|s - y|^p \sqrt{s}} \, ds \\
\varphi_2(y) &= \int_1^\infty \frac{1}{|s - y|^p} \, ds = \int_0^|1 - y - t|^p dt \\
\varphi_3(y) &= \int_{-1}^1 |1 - y - t^2| \, dt \quad \text{for } y > 0
\end{align*} \]

$\varphi_1$ satisfies $\varphi_1(y) = \varphi_1(1 - y)$ and behaves at $y = 0$ and $y = \infty$ like

\[ \varphi_1(y) \sim \begin{cases} 
c_1 y^{(1/p + 1/2) \wedge 0} & \text{if } p \neq 1/2 \\
c_2 y^{-p} & \text{as } y \to \infty
\end{cases} \quad \text{as } y \to 0 \tag{120} \]

$\varphi_2, \varphi_3(y)$ are estimated as

\[ \begin{align*}
\varphi_2(y) &\leq c_3 \begin{cases} 
\epsilon |y - 1|^{-p} & \text{if } |y - 1| > \epsilon \\
\epsilon^2 & \text{if } |y - 1| < \epsilon
\end{cases} \\
\varphi_3(y) &\leq c_4 \begin{cases} 
y^{-1} & \text{if } y > 2 \\
\epsilon^{-p} |y - 1|^{-p} (y^{\alpha q} + 1) & \text{if } 0 < y < 2
\end{cases}
\end{align*} \tag{121} \]

Proof. The identity $\varphi_1(y) = \varphi_1(1 - y)$ is clear. If $p < 1/2$, then $\varphi_1(y)$ has a finite limit when $y \to 0$

\[ \varphi_1(0+) = \int_0^1 \frac{1}{s^p \sqrt{s(1 - s)}} \, ds = \sqrt{\frac{\Gamma(1/2 - p)}{\Gamma(1 - p)}}. \]

If $p > 1/2$, dividing the integral into two parts, we have

\[ \varphi_1(y) = \int_0^{1/2} \frac{1}{|s - y|^p \sqrt{s}} \, ds + \psi(y) \]

\[ = y^{p+1/2} \int_0^{1/(2y)} \frac{1}{|s - 1|^p \sqrt{s(1 - ys)}} \, ds + \psi(y) \]

with

\[ \psi(y) = \int_1^1 \frac{1}{|s - y|^p \sqrt{s(1 - s)}} \, ds. \]

Then, letting $y \to 0$ yields

\[ y^{p-1/2} \varphi_1(y) \to \int_0^\infty \frac{1}{|s - 1|^p \sqrt{s}} \, ds. \]

The asymptotics of $\varphi_1(y)$ at $y = \infty$ is easy to obtain.

A change of variable shows

\[ \varphi_2(y) = |y - 1|^q \begin{cases} 
\int_0^{|1 - y|^{-1} - 1} \frac{1}{|1 - t|^p} dt & \text{if } 0 < y < 1 \\
\int_0^{|1 - y|^{-1} - 1} \frac{1}{|1 + t|^p} dt & \text{if } y > 1
\end{cases} \]

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which leads us easily to the first part of (121). As for \( \varphi_3(y) \) clearly \( \varphi_3(y) \leq c_5y^{-1} \) if \( y > 2 \). Moreover, if \( 0 < y < 2 \) we apply the inequality \( |z| \geq \text{const.} |\text{Re } z|^p |\text{Im } z|^q \) and we have

\[
\varphi_3(y) \leq c_5 |1 - y|^{-p} e^{-q} \int_{-1}^{1} \frac{dt}{|t - y - \alpha|^q}
\]

\[
= c_5 |1 - y|^{-p} e^{-q} y^{-\alpha p} \int_{-y^\omega}^{y^\omega} \frac{ds}{|s - 1|^q}
\]

\[
\leq c_6 |1 - y|^{-p} e^{-q} (y^{\alpha q} \wedge 1),
\]

hence the second estimate of (121) follows. ■

Let \( C \) be the curve \( C_\omega \) of (43) with \( \omega \) satisfying (44), and define \( \mathcal{M}_N \) with respect to this curve \( C \). We prove the approximation of \( m \in \mathcal{M}_N \) by the above \( m^{a,b} \) with the norm \( \| \cdot \|_{n,M} \) for some \( M \leq N \). Set

\[
\alpha = n/2 - 1.
\]

Proof of Lemma 31

Fix \( b \) such that \( b > -\lambda_0 \). For \( m \in \mathcal{M}_N \) define \( m_a = m^{a,b} \) in Lemma 54. Since \( m, m_a - 1 \in \Phi^N_\alpha \), the Cauchy integral formula allows representations

\[
\begin{cases}
  m_e(z) = \frac{1}{2\pi i} \int_{C_\alpha} \frac{m_e(\lambda)}{z-\lambda} \, d\lambda \\
  m_o(z) = 1 + \sqrt{1 - a/z} \frac{1}{2\pi i} \int_{C_\alpha} \frac{m_o(\lambda) - 1}{(z-\lambda)\sqrt{1 - a/\lambda}} \, d\lambda
\end{cases}
\]  

(122)

for \( z \in D_-(=D^-_\alpha) \). In the second identity we have used the fact that \( (m_o(z) - 1)/\sqrt{1 - a/z} \in H(D_-) \).

(122) yields

\[
\begin{cases}
  m_e(z) - m_{a,e}(z) = \frac{1}{2\pi i} \int_{C_a} \frac{m_e(\lambda)}{z-\lambda} \, d\lambda \\
  m_o(z) - m_{a,o}(z) = \sqrt{1 - a/z} \frac{1}{2\pi i} \int_{C_a} \frac{m_o(\lambda) - 1}{(z-\lambda)\sqrt{1 - a/\lambda}} \, d\lambda
\end{cases}
\]

(123)

where \( C_a \) is a curve defined by

\[
C_a = C^1_a + C^2_a
\]

with

\[
\begin{align*}
C^1_a &= \{ x \pm i\omega(x) \mid x \leq a \} \\
C^2_a &= \{ a + it(-a)^{-\alpha} \mid |t| \leq 1 \}
\end{align*}
\]

(see f.5).

We treat only \( m_o(z) \), since \( m_e(z) \) can be discussed similarly and more easily. Since \( m_o - 1 \in \Phi^N_\alpha \), the definition of \( \Phi^N_\alpha \) implies that there exists a rational function \( f \) in \( H(D_+) \) \( (D_+ = D^+_\alpha) \) such that

\[
\int_{C^1} \left| \lambda^N (m_o(\lambda) - 1 - f(\lambda)) \right|^2 |d\lambda| < \infty.
\]  

(124)
Since \( \sqrt{1 - a/z} \) is holomorphic on \( \mathbb{C} \setminus [a, 0] \), (123) turns to
\[
m_o(z) - m_{a,o}(z) = \frac{\sqrt{1 - a/z}}{2\pi i} \int_{C_a} \frac{m_o(\lambda') - 1 - f(\lambda')}{(z - \lambda') \sqrt{1 - a/\lambda'}} d\lambda'.
\] The identity
\[
\sqrt{1 - a/z} = 1 - \int_a^0 \frac{\sqrt{a/\lambda - 1}}{\lambda - z} \frac{d\lambda}{\pi}
\]
together with
\[
(\lambda - z)^{-1} (z - \lambda')^{-1} = (\lambda - \lambda')^{-1} \left( (\lambda - z)^{-1} + (z - \lambda')^{-1} \right)
\]
yields
\[
m_o(z) - m_{a,o}(z) = - \int_a^0 \frac{\sqrt{a/\lambda - 1}}{\lambda - z} \frac{d\lambda}{\pi} \left( \frac{m_o(\lambda') - 1 - f(\lambda')}{\sqrt{1 - a/\lambda'}} \right)
\]
\[
+ \frac{1}{2\pi i} \int_{C_a} \frac{m_o(\lambda) - 1 - f(\lambda)}{z - \lambda} d\lambda.
\] (126)

For \( b_1 > b \) an expansion
\[
\frac{1}{\lambda - z} = - \sum_{k=0}^{M-1} (\lambda - b_1)^k (z - b_1)^{-k-1} + \frac{(\lambda - b_1)^M (z - b_1)^{-M}}{\lambda - z}
\] (127) shows
\[
\int_a^0 \frac{\sqrt{a/\lambda - 1}}{\lambda - z} \frac{d\lambda}{\pi} \frac{1}{2\pi i} \int_{C_a} \frac{m_o(\lambda') - 1 - f(\lambda')}{\sqrt{1 - a/\lambda'}} d\lambda' = f_o^a(z) + \frac{(z - b_1)^{-M}}{\pi} \delta_a(z),
\]
where
\[
\begin{align*}
\delta_a(z) &= \int_a^0 \frac{(\lambda - b_1)^M \sqrt{a/\lambda - 1}}{\lambda - z} \frac{d\lambda}{\pi} \frac{1}{2\pi i} \int_{C_a} \frac{m_o(\lambda') - 1 - f(\lambda')}{\sqrt{1 - a/\lambda'}} d\lambda', \\
f_o^a(z) &= \int_a^0 \frac{(\lambda - b_1)^M \sqrt{a/\lambda - 1}}{\lambda - z} \frac{d\lambda}{\pi} \frac{1}{2\pi i} \int_{C_a} \frac{m_o(\lambda') - 1 - f(\lambda')}{\sqrt{1 - a/\lambda'}} d\lambda'.
\end{align*}
\]
The integer \( M \geq 0 \) is chosen later. We have to show \( \delta_a \to 0 \) in \( L^2(C) \) if \( a \) tends to \(-\infty\). We prove it by decomposing \( \delta_a \). Let \( \delta_a(z) = \delta_a^1(z) + \delta_a^2(z) \) with \( \delta_a^\ell \) defined by integration on \( C_a^j \) for \( j = 1, 2 \) respectively. \( \delta_a^\ell \) can be estimated as follows. For \( \lambda \in [a, 0] \), \( z \in D_- \) we have
\[
|\delta_a^1(z)| \leq \int_a^0 |\lambda|^M \sqrt{a/\lambda - 1} \frac{d\lambda}{\pi} \int_{C_a^1} \left| \frac{m_o(\lambda') - 1 - f(\lambda')}{\sqrt{1 - a/\lambda'}} \right| d\lambda' \leq c_1 \int_a^0 |\lambda|^M \sqrt{a/\lambda - 1} \frac{d\lambda}{\pi} \int_{C_a^1} \frac{|\lambda'|^{-2N}}{(\sqrt{1 - a/\lambda'})^2} d\lambda'
\] 64
with \( c_1 = \| \lambda^N (m_o - 1 - f) \| / 2\pi^2 \). Change of variables \( \lambda = as, \ \lambda' = at + i (-at)^{-\alpha} \) shows

\[
\int_0^a |\lambda|^M \sqrt{a/\lambda - 1} |\lambda|^N \sqrt{1 - a/\lambda'} \ d\lambda \int_{C_2} \frac{|\lambda'|^{-2N} |d\lambda'|}{(\lambda - \lambda') \sqrt{1 - a/\lambda'}} = |a|^M \int_0^1 s^M \sqrt{1/s - 1} T(s) ds \int_{C_2} \frac{I(s)}{|s - a^{-1} z|},
\]

where

\[
I(s) = 2 \int_1^\infty \frac{[(as - (at + i (-at)^{-\alpha}))^2 - 1 + \alpha i (-at)^{-\alpha - 1} |a|]}{|(as - (at + i (-at)^{-\alpha})) \sqrt{1 - a/ (at + i (-at)^{-\alpha})}|^2} dt.
\]

For simplicity of the notations set \( \epsilon = 2^{-\alpha} (-a)^{-\alpha - 1} \). Then, with some constants \( c_2, c_3 \) we have

\[
I(s) \leq |a|^{-2N} \left( c_2 \int_1^2 \frac{t^{-2N}}{[(s - t + i\epsilon) \sqrt{1 - 1/(t - i\epsilon)}]} dt + c_3 \right).
\]

Since \( |s - t + i\epsilon| \geq 1 - s \) and \( t^{-2N} |t - i\epsilon| \leq 3 \) if \( |a| \) is sufficiently large, \( I(s) \) is dominated as

\[
I(s) \leq |a|^{-2N} \left( 3c_2 (1 - s)^{-2} \int_1^2 \frac{1}{|t - 1 - i\epsilon|} dt + c_3 \right) \leq c_4 |a|^{-2N} (1 - s)^{-2} \log (1 + \epsilon^{-1})
\]

with a constant \( c_4 \). Therefore

\[
|\delta_0(x)| \leq c_1 \sqrt{c_2} |a|^{-N} |a|^M \sqrt{\log (1 + \epsilon^{-1})} \int_0^1 \frac{ds}{|s - a^{-1} z| \sqrt{s (1 - s)}}
\]

holds. Noting that for \( p, q > 0 \) satisfying \( p + q = 1 \) there exists a constant \( c_q \) such that \( |z| \geq c_q \text{Re } z|^p \text{Im } z|^q \) holds for any \( z \in \mathbb{C} \), we have

\[
\int_C \left( \int_0^1 \frac{ds}{|s - a^{-1} z| \sqrt{s (1 - s)}} \right)^2 \ |dz| \leq c_q^{-2} |a|^{2q(1+\alpha)} \int_0^1 |x|^{2aq} \varphi_1 (a^{-1} x)^2 \ dx
\]

with \( \varphi_1 \) in Lemma 55. Here we have neglected the part \( \{ x \pm i\omega (x) \}_{-1 \leq x \leq b} \), which is easily treated. Since

\[
\varphi_1 (y) \sim c_5 y^{-p} \text{ as } y \to \infty
\]

is valid if \( 0 < p < 1 \), hence, if \( p, q \) satisfy is \( 2aq - 2p < -1 \), namely \( p > (2\alpha + 1) / (2\alpha + 2) \), it holds that

\[
\int_{-\infty}^1 |x|^{2aq} \varphi_1 (a^{-1} x)^2 \ dx \leq c_6 |a|^{2aq+1},
\]

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and
\[
\int_C |\delta_1^n(z)|^2 dz \leq c_5 |a|^{2\alpha q + 1} |a|^{-2N} |a|^{2M} |a|^{2q(1+\alpha)} \log (1 + \epsilon^{-1})
\]
\[
= c_5 |a|^{2(2\alpha + 1)q - 2(N - M) + 1} \log (1 + 2\alpha |a|^\alpha + 1),
\]
which converges to 0 as \( a \to -\infty \) for \( M \leq N - 1 \) if \( q \) is sufficiently small.

When we estimate \( \delta_2^n \), the decay on \( C \) of \( m_\alpha - 1 - f \) has nothing to do with the estimate on \( C_2^n \), which causes difficulty. Set \( \lambda = as, \lambda' = a + it (a)^{-\alpha} \). Then
\[
\delta_2^n(z) = I_1(z, a) - I_2(z, a),
\]
where
\[
\begin{align*}
I_1(z, a) &= a^M \int_0^1 \frac{(s - a^{-1}b) \sqrt{1/s - 1}}{s - a^{-1}z} ds \\
&\quad \times \frac{\sqrt{\epsilon}}{2\pi} \int_{-1}^1 \frac{1}{(s - 1 + it\lambda' \sqrt{-it})} dt \\
I_2(z, a) &= a^M \int_0^1 \frac{(s - a^{-1}b) \sqrt{1/s - 1}}{s - a^{-1}z} ds \\
&\quad \times \frac{\sqrt{\epsilon}}{2\pi} \int_{-1}^1 (1 + f(a + it(-a)^{-\alpha})) dt,
\end{align*}
\]
with \( \epsilon = (-a)^{-\alpha - 1} \). Since \( f \) is a rational function having no poles in \( D_+(= D_+^\lambda) \) and \( f \in L^2(C_{-\infty}) \), \( f \) is of order \( O\left(|z|^{-1}\right) \). Hence there exists a constant \( c_6 \) such that \( |1 + f(a + it(-a)^{-\alpha})| \leq c_6 \). First we estimate \( I_2(z, a) \).

\[
|I_2(z, a)| \leq c_6 \sqrt{\epsilon} |a|^M \int_0^1 \frac{\sqrt{1/s - 1}}{|s - a^{-1}z|} ds \int_0^1 \frac{dt}{|s - 1 + it\lambda| \sqrt{t}}
\]
The estimate
\[
\int_0^1 \frac{1}{|s - 1 + it\lambda| \sqrt{t}} dt \leq c_7 (1 - s)^{-1/2} \epsilon^{-1/2} \left( \epsilon^{1/2} (1 - s)^{-1/2} \wedge 1 \right) \quad (128)
\]
implies
\[
|I_2(z, a)| \leq c_8 |a|^M \int_0^1 \frac{\epsilon^{1/2} (1 - s)^{-1/2}}{|s - a^{-1}z| \sqrt{s}} ds \\
\leq c_8 |a|^M \sqrt{\epsilon} \int_0^1 \frac{ds}{|s - a^{-1}z| \sqrt{s(1 - s)}} + c_8 |a|^M \int_{1-\epsilon}^1 \frac{1}{|s - a^{-1}z|} ds \\
\leq c_8 \sqrt{\epsilon} |a|^{M+q} |x|^{\alpha q} \varphi_1(a^{-1}x) + c_8 |a|^{M+q} |x|^{\alpha q} \varphi_2(a^{-1}x), \quad (129)
\]
where we have used the inequality $|z| \geq c_q |\text{Re} z|^p |\text{Im} z|^q$ for $p, q$ satisfying $p + q = 1$. Therefore,

$$
\int_C |I_2 (z, a)|^2 \, |dz|
\leq c_3 |a|^{2M+2q} \left( \epsilon \int_{-\infty}^{-1} |x|^{2aq} \varphi_1 (a^{-1} x)^2 \, dx + \int_{-\infty}^{-1} |x|^{2aq} \varphi_2 (a^{-1} x)^2 \, dx \right) + \int_{-a^{-1}}^{a^{-1}} y^{2aq} \varphi_1 (y)^2 \, dy + c_3 |a|^{2M+2q} \int_{-a^{-1}}^{a^{-1}} y^{2aq} \varphi_2 (y)^2 \, dy
$$

(130)

holds. The first integral converges if $2aq - 2p < -1$. As for the second term (121) implies

$$
\int_{-a^{-1}}^{a^{-1}} y^{2aq} \varphi_2 (y)^2 \, dy
\leq c_4 \epsilon^2 \int_{-a^{-1}}^{a^{-1}} y^{2aq} |y - 1|^{-2p} \, dy + c_4 \epsilon^2 \int_{1-a^{-1}}^{1+a^{-1}} y^{2aq} \, dy + c_4 \epsilon^2 \int_{1+a^{-1}}^{\infty} y^{2aq} |y - 1|^{-2p} \, dy
\leq c_1 \epsilon^{2q+1}.
$$

Therefore, $\int_C |I_2 (z, a)|^2 \, |dz|$ is dominated by constant times of

$$
|a|^{2M+2q+2aq+1-(a+1) + |a|^{2M+2q+2aq+1-(2q+1)(a+1)} = |a|^{2M+(1-a)+1-2q} + |a|^{2M-a}.
$$

Hence, if $2M - (a+1) (1-2q) < 0$, then the above term tends to 0 as $a \to -\infty$. If $2M < a$, then one can choose suitable $p, q$ satisfying two conditions $2aq - 2p < -1$ and $2M + (a+1)(1-2q) < 0$ simultaneously.

The rest of the proof is to show $I_1 (z, a) \to 0$ in $L^2 (C)$. First note that $2\sqrt{-\bar{z}m_\alpha (-z)}$ is a Herglotz function which satisfies

$$
\begin{cases}
2\sqrt{-\bar{z}m_\alpha (-z)} \sim 2\sqrt{-z} \quad \text{as} \quad z \to \infty \\
2\sqrt{-\bar{z}m_\alpha (-z)} \quad \text{is analytic on} \quad (-\infty, \lambda_0) \quad \text{and takes real values}
\end{cases}
$$

Therefore, there exists a measure $\sigma$ on $[\lambda_0, \infty)$ such that

$$
2\sqrt{-\bar{z}m_\alpha (-z)} = \int_{\lambda_0}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sigma \, (d\xi) = \int_{\lambda_0}^{\infty} \frac{1 + \xi z}{\xi - z} \nu \, (d\xi)
$$

with $\nu (d\xi) = \sigma (d\xi) / (\xi^2 + 1)$. $\nu$ is a measure on $[\lambda_0, \infty)$ with finite total mass. Keeping this representation in mind we set

$$
I_1 (z, \xi, a) = a^M \int_0^1 \frac{1}{s - a^{-1} b}_1 \frac{M}{M} \frac{1/\sqrt{s - 1}}{\sqrt{s - 1}} \, ds \frac{1}{2\pi i} \int_{-1}^{1} \frac{1 - \text{ic} \, dt}{s - 1 + \text{ic}} \, \sqrt{-it}
\times \frac{1 + a \xi (1 - \text{ic})}{2 \sqrt{a (1 - \text{ic}) (\xi - a (1 - \text{ic}))}}.
$$

Then, $I_1 (z, \xi, a)$ is dominated as

$$
|I_1 (z, \xi, a)|
\leq c_{10} |a|^{M-1/2} \sqrt{\epsilon} \int_0^1 \frac{1}{|s - a^{-1} b|} \frac{1/\sqrt{s - 1}}{\sqrt{s - 1}} \, ds \int_{-1}^{1} \frac{|1 + a \xi (1 - \text{ic})|}{|s - 1 + \text{ic}||\xi - a (1 - \text{ic})| \sqrt{|t|}}. \quad (131)
$$
Set

\[ \psi (\xi, a) = \sup_{|t| \leq 1} \frac{|1 + a \xi (1 - i t)|}{|\xi - a (1 - i t)|}. \]

Then, it holds that

\[
\psi (\xi, a) = \frac{|a^{-1} + \xi|}{|\xi a^{-1} - 1|} I_{|a^{-1} \xi^{-1} + 1| \geq |\xi a^{-1} - 1|} + \frac{|a^{-1} + \xi - i \xi|}{|\xi a^{-1} - 1 + i t|} I_{|a^{-1} \xi^{-1} + 1| < |\xi a^{-1} - 1|},
\]

and we have

\[
|I_1 (z, \xi, a)| \leq c_{10} |a|^{M-1/2} \sqrt{\psi (\xi, a)} \int_0^1 \frac{t}{|s - a^{-1} z|} \left| \int_{-1}^1 \frac{dt}{|s - 1 + i t| \sqrt{|t|}} \right|
\]

Then, it holds that

\[
|I_1 (z, \xi, a)| \leq c_{11} |a|^{M-1/2} \psi (\xi, a) \int_0^1 \frac{t}{|s - a^{-1} z|} \left( \int_{-1}^1 \frac{dt}{|s - 1 + i t| \sqrt{|t|}} \right)
\]

Now the norm of \( I_1 (z, a) \) can be calculated as

\[
\int_C |I_1 (z, a)|^2 |dz| = \int_C |dz| \left| \int_0^1 I_1 (z, \xi, a) \nu (d\xi) \right|^2 \leq c_{13} |a|^{-1} \left( \int_0^1 \psi (\xi, a) \nu (d\xi) \right)^2 \left( |a|^{2M+2q} \int_{-\infty}^1 \left| x \right|^{2\alpha q} \left( t_1 (a^{-1} x) + t_2 (a^{-1} x) \right) dx \right).
\]

The second integral was already estimated in (119) and is dominated by \( |a|^{2M-\alpha} \).

The first integral is calculated by (132), that is

\[
\int_{\lambda_0}^\infty \psi (\xi, a) \nu (d\xi) = \int_{|a^{-1} \xi^{-1} + 1| \geq |\xi a^{-1} - 1|} \frac{|a^{-1} + \xi|}{|\xi a^{-1} - 1|} \nu (d\xi) + \int_{|a^{-1} \xi^{-1} + 1| < |\xi a^{-1} - 1|} \frac{|a^{-1} + \xi - i \xi|}{|\xi a^{-1} - 1 + i t|} \nu (d\xi)
\]

\[
= \int_a^\infty \left[ \frac{|a^{-1} + \xi|}{|\xi a^{-1} - 1|} \nu (d\xi) + \int_{|\xi a^{-1} - 1 + i t|}^{a^{-1} \xi^{-1} + 1} \frac{|a^{-1} + \xi - i \xi|}{|\xi a^{-1} - 1 + i t|} \nu (d\xi) \right] + \int_{-\infty}^{a^{-1} \xi^{-1} + 1} \frac{|a^{-1} + \xi - i \xi|}{|\xi a^{-1} - 1 + i t|} \nu (d\xi)
\]

\[
= \int_{|a^{-1} \xi^{-1} + 1| \geq |\xi a^{-1} - 1|} \frac{|a^{-1} + \xi|}{|\xi a^{-1} - 1|} \nu (d\xi) + \int_{|\xi a^{-1} - 1 + i t|}^{a^{-1} \xi^{-1} + 1} \frac{|a^{-1} + \xi - i \xi|}{|\xi a^{-1} - 1 + i t|} \nu (d\xi) + \int_{|a^{-1} \xi^{-1} + 1| < |\xi a^{-1} - 1|} \frac{|a^{-1} + \xi|}{|\xi a^{-1} - 1 + i t|} \nu (d\xi).
\]

The sum of the first two terms tends to \( \int_{\lambda_0}^0 |\xi| \nu (d\xi) \) as \( a \to -\infty \). To estimate the last term we use

\[
\sup_{\xi \geq |a^{-1}| \xi \geq |a^{-1} \xi^{-1} + 1|} \frac{|a^{-1} + \xi - i \xi|}{|\xi a^{-1} - 1 + i t|} = \sqrt{1 + e^2} |a|.
\]

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Consequently we have
\[
\int_C |I_1(z,a)|^2 |dz| \leq c_{14} |a|^{2M+2-(\alpha+1)(1-2q)},
\]
and, if \(2M < \alpha-1\) is satisfied, then the above integral converges to 0 as \(a \to -\infty\).

**The second term** of (126) is treated similarly but more easily. Applying the expansion (127) to this term we obtain
\[
\begin{aligned}
&\frac{1}{2\pi i} \int_{C_a} \frac{m_o(\lambda) - 1 - f(\lambda)}{z - \lambda} d\lambda \\
&= - \sum_{k=0}^{M-1} (z - b_1)^{-k-1} \frac{1}{2\pi i} \int_{C_a} (\lambda - b_1)^k (m_o(\lambda) - 1 - f(\lambda)) d\lambda \\
&+ (z - b_1)^{-M} \frac{1}{2\pi i} \int_{C_a} (\lambda - b_1)^M (m_o(\lambda) - 1 - f(\lambda)) d\lambda.
\end{aligned}
\]

Then, the last term is divided into two terms:
\[
\begin{aligned}
&\frac{1}{2\pi i} \int_{C_a} \frac{(\lambda - b_1)^M (m_o(\lambda) - 1 - f(\lambda))}{\lambda - z} d\lambda = \delta^3_a(z) + \delta^4_a(z),
\end{aligned}
\]
where
\[
\begin{align*}
\delta^3_a(z) &= \frac{1}{2\pi i} \int_{C_a} \frac{(\lambda - b_1)^M (m_o(\lambda) - 1 - f(\lambda))}{\lambda - z} d\lambda, \\
\delta^4_a(z) &= \frac{1}{2\pi i} \int_{C_a} \frac{(\lambda - b_1)^M (m_o(\lambda) - 1 - f(\lambda))}{\lambda - z} d\lambda.
\end{align*}
\]
In this case \(\delta^3_a(z)\) can be estimated by applying the boundedness of the Cauchy integral on a curve with Lipschitz continuity (see ([30]), ([31])), which shows that there exists a constant \(c_{15}\) such that
\[
\int_C |\delta^3_a(z)|^2 |dz| \leq c_{15} \int_C \left| (\lambda - b_1)^M (m_o(\lambda) - 1 - f(\lambda)) I_{C_a}^\lambda (\lambda) \right|^2 |d\lambda|
\]
\[
= c_{15} \int_{C_a} \left| (\lambda - b_1)^M (m_o(\lambda) - 1 - f(\lambda)) \right|^2 |d\lambda|,
\]
which clearly tends to 0 as \(a \to -\infty\).

**Proof.** On the other hand, we have to employ the method used for \(\delta^2_a(z)\) to treat \(\delta^4_a(z)\). A similar calculation to that of \(\delta^2_a(z)\) we arrive at
\[
\int_C |\delta^4_a(z)|^2 |dz| \leq c_{16} |a|^{2M} \left( \int_{\lambda_0}^\infty \psi(\xi,\alpha) \nu(\xi) \right)^2 \int_{-a-1}^\infty \varphi_3(y)^2 dy.
\]
Then, applying the second estimate of (121) we have
\[
\begin{aligned}
\int_{-a-1}^\infty \varphi_3(y)^2 dy &\leq c_{17} \int_{-a}^\infty y^{-2} dy + c_{18} \epsilon^{-2q} \int_0^2 |y - 1|^{-2q} (y^{2q} \wedge 1) dy \\
&\leq c_{19} \epsilon^{-2q},
\end{aligned}
\]
if \(2p < 1\), and due to (133), (134) an estimate
\[
\int_C |\delta_\alpha^4(z)|^2 |dz| \leq c_{20} |a|^{2M+2p} \epsilon^{2-2q} = c_{20} |a|^{2M+2-2p(1+\alpha)}
\]
follows. Therefore, if \(2M + 2 - 2p(1 + \alpha) < 0\), it holds that \(\delta_\alpha^4 \to 0\) in \(L^2(C)\) as \(a \to -\infty\). Choosing \(p\) sufficiently close to \(1/2\), we see that \(2M < \alpha - 1\) is sufficient. Setting \(N' = M\) completes the proof. □

### 8.4 Estimate of relevant integrals

**Lemma 56** Let \(C\) be a curve \(C_\omega\) of (43) with \(\omega(x) \sim (-x)^{-n/2+1}\). For \(\beta\), \(M \geq 0\) satisfying \(\beta - M < 1/2\) and an odd real polynomial \(h\) of degree \(n\)
\[
\int_C \left| \frac{z^\beta e^{h(\sqrt{z})} - \lambda^\beta e^{h(\sqrt{\lambda})}}{z - \lambda} \right|^2 |dz| \leq c \begin{cases} |\lambda|^{n-1-2M+2\beta} & \text{if } M \leq (n+1)/2 \\ |\lambda|^{2\beta-2} & \text{if } M > (n+1)/2 \end{cases}
\]
holds for \(z \in C\) with a constant \(c > 0\).

**Proof.** The integrand is decomposed into
\[
\frac{z^\beta e^{h(\sqrt{z})} - \lambda^\beta e^{h(\sqrt{\lambda})}}{z - \lambda} = \frac{e^{h(\sqrt{z})} - e^{h(\sqrt{\lambda})}}{z - \lambda} z^{-M+\beta} + \frac{z^\beta - \lambda^\beta}{z - \lambda} z^{-M} e^{h(\sqrt{\lambda})}.
\]
The first term is estimated as follows. To dominate
\[
\int_C \left| z^{-M+\beta} \left( \frac{e^{h(\sqrt{z})} - e^{h(\sqrt{\lambda})}}{z - \lambda} \right)^2 |dz| = \int_C \left| \frac{e^{h(\sqrt{z})} - e^{h(\sqrt{\lambda})}}{z - \lambda} \right|^2 |z|^{-2M+2\beta} |dz|
\]
we replace \(C, z, \lambda\) by \([1, \infty), -x, -y\) respectively and compute
\[
\int_1^\infty \left| \frac{e^{ix^{n/2}} - e^{iy^{n/2}}}{x - y} \right|^2 y^{-2M+2\beta} dy = I_1(x) + I_2(x)
\]
with
\[
\begin{align*}
I_1(x) &= \int_{|x-y|<x/2, y>1} \left| \frac{e^{ix^{n/2}} - e^{iy^{n/2}}}{x - y} \right|^2 y^{-2M+2\beta} dy \\
I_2(x) &= \int_{|x-y|>x/2, y>1} \left| \frac{e^{ix^{n/2}} - e^{iy^{n/2}}}{x - y} \right|^2 y^{-2M+2\beta} dy
\end{align*}
\]
Then
\[
I_1(x) \leq c_1 \int_{|x-y|<x/2} \left| \frac{x^{n/2} - y^{n/2}}{x - y} \right|^2 y^{-2M+2\beta} dy
= c_1 x^{n-2-2M+2\beta+1} \int_{|t|<1/2} \left| \frac{1 - t^{n/2}}{1 - t} \right|^2 t^{-2M+2\beta} dt \leq c_2 x^{n-1-2M+2\beta},
\]
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and

\[ I_2(x) \leq c_2 \int_{|x-y| > x/2, y > 1} (x - y)^{-2} y^{-2M + 2\beta} dy \]

\[ = c_2 x^{-1 - 2M + 2\beta} \int_{|1-t| > 1/2, t > x^{-1}} (1 - t)^{-2} t^{-2M + 2\beta} dt \]

\[ \leq c_3 \begin{cases} 
  x^{-2} & \text{if } 2(M - \beta) > 1 \\
  x^{-2} \log x & \text{if } 2(M - \beta) = 1 \\
  x^{-1 - 2M + 2\beta} & \text{if } 2(M - \beta) < 1
\end{cases} \]

On the other hand, since \( e^{b(\sqrt{x})} \) is bounded, the second term of (135) is equivalent to

\[ \int_1^\infty \left| \frac{x^\beta - y^\beta}{x - y} \right|^2 y^{-2M} dy, \]

which is equal to

\[ x^{2\beta - 2M - 1} \int_{x-1}^\infty \left| \frac{1-t^\beta}{1-t} \right|^2 t^{-2M} dt \sim c_4 \begin{cases} 
  x^{2\beta - 2} & \text{if } 2M > 1 \\
  x^{2\beta - 2} \log x & \text{if } 2M = 1 \\
  x^{2\beta - 2M - 1} & \text{if } 2M < 1
\end{cases} \]

as \( x \to \infty \). One can see that (136) always dominates \( I_2(x) \), and comparing (136) with \( I_1(x) \) we have the lemma.  

References


