Gaussian integral representation of determinant and KdV equation

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**Gaussian integral representation of determinant**  It is well known that Gaussian integral with a positive definite matrix $\Sigma$ of size $n$ has an identity

$$\frac{1}{\sqrt{\det (2\pi \Sigma)}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\Sigma^{-1}\xi,\xi)} d\xi = 1.$$  

This identity can be shown by diagonalization of $\Sigma$, and gives an expression of $\det \Sigma$ as

$$\left(\det \Sigma\right)^{1/2} = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\Sigma^{-1}\xi,\xi)} d\xi.$$  

Replacing $\Sigma$ by $\Sigma^{-1}$, we have

$$\left(\det \Sigma\right)^{-1/2} = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\Sigma\xi,\xi)} d\xi. \quad (1)$$  

This kind of integral representation is effectively used especially in mathematical physics, and is an important tool when we need to expand the size of matrices to infinity.

**KdV equation**  KdV equation was proposed by British scientists in 19th century to describe the motion of wave travelling on shallow waters, and takes a form of

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}.$$  

Several special solutions to this equation were known, however, in 1960s the discovery of its relationship with eigenvalues of 1-dimensional Schrödinger operators was a trigger of great development of theory of integrable systems. In the course of its evolution the following solution known as $n$-soliton was obtained.

$$u(t, x) = -2\frac{\partial^2}{\partial x^2} \log \det (I + A(t, x)),$$

where, for $m_i, \eta_i > 0$

$$A(t, x) = \left(\frac{\sqrt{m_i}m_j}{\eta_i + \eta_j} e^{-(\eta_i + \eta_j)x + 4(\eta_i^3 + \eta_j^3)t}\right)_{1 \leq i, j \leq n}.$$  

The entries of this matrix have an expression

$$\frac{\sqrt{m_i}m_j}{\eta_i + \eta_j} e^{-(\eta_i + \eta_j)x + 4(\eta_i^3 + \eta_j^3)t} = \sqrt{m_i}\sqrt{m_j} e^{4\eta_i^3 t} e^{4\eta_j^3 t} \int_x^\infty e^{-\eta_i y} e^{-\eta_j y} dy, \quad (2)$$
hence $A(t, x)$ is positive definite. (2) gives
\[
(A(t, x)\xi, \xi) = \int_x^\infty \left( \sum_{i=1}^n \sqrt{m_i} e^{4\eta_i t} e^{-\eta_i^2 y} \right)^2 dy,
\]
and (1) shows a representation
\[
\det (I + A(t, x))^{-\frac{1}{2}} = \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \int_x^\infty \left( \sum_{i=1}^n \sqrt{m_i} e^{4\eta_i t} e^{-\eta_i^2 y} \right)^2 dy \right) e^{-\frac{1}{2} \|\xi\|^2} d\xi.
\]

\section*{Gaussian processes}
Knowledge of probability theory helps us to understand the function in (3) as a probability density of i.i.d. random variables $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ with standard normal distribution. Now by using these random variables $\{\xi_i\}$ define
\[
X(t, x) = \sum_{i=1}^n \sqrt{m_i} e^{4\eta_i t} e^{-\eta_i x} \xi_i.
\]
Then $X(t, x)$ is a 2-parameter Gaussian processes, and (3) turns to
\[
\det (I + A(t, x))^{-\frac{1}{2}} = \mathbb{E} \left( \exp \left( -\frac{1}{2} \int_x^\infty X(t, y)^2 dy \right) \right),
\]
where $\mathbb{E}$ denotes the expectation. Consequently the $n$-soliton solution to KdV equation can be represented as
\[
u(t, x) = 4 \frac{\partial^2}{\partial x^2} \log \mathbb{E} \left( \exp \left( -\frac{1}{2} \int_x^\infty X(t, y)^2 dy \right) \right).
\]

\section*{Generalization}
Gesztesy-Karwowsky obtained $\infty$–soliton solutions by employing a Fredholm determinant instead of Gaussian integral. However, this expression makes it possible to let $n$ tend to infinity easier. A little knowledge of probability theory suggests us to look (4) as follows: For a measure $m$ on the right half line $(0, \infty)$ let $M(d\eta)$ be a Gaussian random measure with covariance measure $m$. For a discrete measure defined by
\[
m(d\eta) = \sum_{i=1}^n m_i \delta_{\{\eta_i\}}(d\eta)
\]
$M$ is
\[
M(d\eta) = \sum_{i=1}^n \sqrt{m_i} \xi_i \delta_{\{\eta_i\}}(d\eta).
\]
Define a Gaussian process $X(t, x)$ by a Wiener’s integral
\[
X(t, x) = \int_0^\infty e^{\eta^2 t - \eta x} M(d\eta).
\]
A sufficient condition to guarantee the convergence of this integral is
\[
\int_0^\infty e^{\eta t} m(d\eta) < \infty
\]
for very $t > 0$. Then, (5) gives a solution to KdV equation. If the support of $m$ is an infinity set, then this solution can be called as an $\infty$–soliton solution.
Discussion  The Gaussian process \(X(t,x)\) satisfies a linear equation

\[
\left( \frac{\partial}{\partial t} + 4 \frac{\partial^3}{\partial x^3} \right) X(t,x) = 0. \tag{6}
\]

A relationship between a linear equation and (non-linear) KdV equation was already known by S.A.Marchenko. He derived solutions to KdV equation as a contraction of Hilbert space valued linear equation instead of using Gaussian processes.

Since this generalized solution decays as \(x \to \infty\), (5) can not give any periodic nor almost periodic solution, even though we use a general \(m\). Although a renormalization procedure enables us to extend the class of solutions slightly, to obtain any periodic solution is impossible. It would be desirable to clarify the class of solutions represented by (5). On the other hand, it seems that this linear equation (6) for a Gaussian process is not sufficient to obtain a solution to KdV equation by (5). The initial function should be of a special form. Moreover, the proof of (5)’s being a solution to KdV equation should be given by a more direct approach, such as a certain calculus among Gaussian processes or Wiener’s integral of degree 2. Probably the algebraic method invented by Date-Jimbo-Miwa would be a hint to establish such a calculus.