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Optimal Sales Schemes against Interdependent Buyers

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Abstract

This paper studies a monopoly pricing problem when the seller can choose the timing of a trade with each buyer, and a buyer's valuation of the seller's good is the weighted sum of his and other buyers' private signals. We show that it is optimal for the seller to employ a *sequential scheme* which trades with one buyer at a time, and hence allows each buyer to observe the outcomes of all preceding transactions. We also identify conditions under which the seller optimally trades with the buyers in the increasing order of the weights they place on other buyers' signals.
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When a seller of a good trades with multiple buyers, he often employs a dynamic sales strategy. That is, instead of serving all the buyers at once, the seller partitions them into smaller markets and supplies each market at a different timing. Examples abound in the entertainment industry where the sellers of books, video games, and movies introduce their products in one market and move on to another after generating hype in the first market. For example, on Sony's announcement on the release of its PS3 game console, a BBC article remarks that "[n]ormally Sony staggers the release of a new console, releasing in Japan and America, with Europe coming a belated third."[1] We also observe similar sales strategies used in automobile and electronics industries.[2]

There is perhaps more than one reason why a seller employs such a dynamic sales strategy. For example, the seller may engage in staggered sales simply because of a constraint on logistics and other physical constraints.[3] In many cases, however, we believe that it is based on strategic motives. For example, in his classical textbook on marketing, Philip Kotler (1988, ch. 14) states that a firm of a new product should choose a particular subset of consumers as first targets, noting that those "[e]arly adopters tend to be opinion leaders and helpful in "advertising" the new product to other potential buyers." That is, a good sales strategy should use the adoption decisions of a small group of consumers with certain characteristics as a signal to other consumers.

In this paper, we explore the possibility of a dynamic sales strategy when there is interdependence among buyers' valuations. More specifically, when buyers' valuations of the seller's good are determined in part by the publicly observable behavior of other buyers, we analyze whether the seller is better off trading with different buyers at different timings. When successful, such a trading strategy can create a success-breed-success process: successful transactions with the initial set of buyers raise the valuations of the next line of buyers, success with the latter raises even further the valuation of the buyers to follow, and so on. Once in such a cycle, the seller can continually increase his offer price and raise more revenue than from static

[1] "PlayStation 3 Euro launch delayed," BBC News, September 6, 2006. It was announced that PS3 is launched on Nov 11, 2006 in Japan, Nov 17 in the U.S. and March 30, 2007 in Europe. A similar staggering strategy was observed for the Nintendo Wii.

[2] One recent example is Toyota's introduction of the Lexus brand to Japan after its well-publicized success in the U.S.

[3] For example, translation is required before best-sellers in English are released in non English countries, producers of electronic devices want to identify program bugs before a launch at a wider scale, etc.

sales. Of course, the seller adopting such a scheme also faces the risk of a downward spiral where a failure in the initial markets leads to a sequence of failures in subsequent markets.

In our model of dynamic trading, a seller faces multiple buyers each endowed with private information about the seller's good. Each buyer demands one unit of the good, which is produced at no cost to the seller. The private signals are independent across buyers and a buyer's valuation of the seller's good is a weighted sum of all buyers' signals. As in the classical monopoly pricing problem, the seller's trading with each buyer takes the form of price posting. The outcomes of transactions are publicly observable to subsequent buyers, and used to update the expected value of the good to them. Each buyer meets the seller once and leaves the market after accepting or rejecting the offer.

The nature of the problem can be best illustrated in a model where there are only two buyers. The seller can either trade with both at once or trade with one of them first and the other next. In the first scheme, referred to here as a *simultaneous scheme,* the seller provides the buyers no opportunity to learn about each other's private signals. In other words, each buyer must make a purchasing decision only on his own signal. In the second scheme, referred to as a *sequential scheme,* the seller allows the second buyer to infer the private signal of his predecessor: Acceptance by the first buyer raises the second buyer's valuation, while rejection lowers it. Note that the exact amount by which the second buyer's valuation changes depends on the level of the price offer to the first buyer: If the first buyer accepts a high price, then the increase in the second buyer's valuation will be large, while if the first buyer accepts a low price, then the increase will be small. In this sense, the seller should choose his price offer in stage 1 so as to balance the rent to extract from buyer 1 and the information to reveal to buyer 2. If the two buyers are not ex ante identical, then the seller must also choose which buyer to serve first. With three or more buyers, the seller's problem is similar but significantly more complex. First, besides sequential and simultaneous schemes, there are a number of intermediate schemes. Second, the choice of buyers at each stage can be contingent on the history of transactions. For example, the buyer(s) with whom the seller may wish to trade in stage 2 may be different depending on the outcome of transaction in stage 1.

The first main conclusion of the paper is that it is optimal for the seller to employ a sequential scheme. The conclusion is based on the construction of a sequential scheme that replicates any given non sequential scheme. In the two-buyer model, for example, suppose that the seller originally employs a simultaneous scheme which offers price x_1 to buyer 1 and x_2 to buyer 2. The alternative sequential scheme offers x_1 to buyer 1 in period 1, and makes contingent offers to buyer 2 in period 2. In particular, the offer to buyer 2 is higher than x_2 when 1 accepts his offer and lower otherwise. The adjustment in the offer to buyer 2 is such that he will accept it with exactly the same probability as he would accept x_2 under the original scheme. Those contingent offers yield the same expected revenue as x_2, and we can also show that even if there are more buyers after buyer 2, their valuations are not affected by this change in the sales scheme and hence the seller's revenue from them can be made unchanged as well.

Given the optimality of a sequential scheme, the second question we address is on the optimal ordering of buyers. For this, we suppose that the buyers' private signals have an identical distribution, and that their valuation equals the own signal plus some constant times the sum of all other buyers' signals. The constant is the unique source of ex ante heterogeneity among the buyers and called the *dependence weight* as it measures how dependent the buyer is on others' information. We look for conditions under which the optimal sales scheme trades in an increasing order of the dependence weights: The first buyer has the smallest weight, the second buyer has the second smallest weight, and so on. When the buyers' private signals have a uniform distribution, we show that a sequential scheme with the monotone ordering is optimal if those weights are similar in size. When the dependence weights have an increasing order, the buyers who are more heavily influenced by public information are given a chance to observe more information. From the perspective of classical monopoly theory, this scheme presents an efficient way to extract the buyers' informational rents since given any public information, the reduction in the rents is higher for a buyer with a larger dependence weight.

The present paper is closely related to the models of social learning and monopoly pricing in sequential sales problems. The models of social learning as studied by Sushil Bikhchandani, David Hirschleifer and Ivo Welch (1992), and Abhijit Banerjee (1992) suppose that infinitely many agents make sequential decisions when they are ex ante identical but have correlated private signals about the underlying state of the world. These models may be interpreted as describing the behavior of buyers who all receive the same take-it-or-leave-it offers from a seller. In contrast, the sequential sales models of Marco Ottaviani (1999), Christophe P. Chamley (2004, Chap. 4), and Subir Bose *et al.* (2006, 2007) suppose that the seller controls the price offered to each buyer. These papers study how optimal pricing by the seller affects the buyers' learning about the true value of the good. Bose *et al.* (2006) for example identifies the range of prior beliefs that allow for complete learning by the buyers. Daniel Sgroi (2002) studies a dual problem of a monopolist under social learning when the seller, who knows the quality of his product and offers a fixed price, partially controls public information by choosing the number of buyers to serve in period 1.[4] The framework of the present paper is different from those of the above in that private signals are independent, valuations are interdependent rather than common, and the buyers are ex ante heterogeneous. The independence of private signals simplifies the analysis significantly and allows us to ignore the seller's learning problem which is central to the analysis of many of the earlier models. The buyer heterogeneity raises the new question of buyer ordering, and provides one possible explanation for the frequent occurrence of consumption fads in some markets not explored by the existing models.

By adopting the sequential scheme, the seller is committed to publicly revealing information about all past transactions to every buyer. Our finding hence parallels the well-known *linkage principle* in auction theory (Paul R. Milgrom and Robert J. Weber (1982)), which states that the auctioneer's expected revenue is maximized when he commits to fully revealing his private information provided that the bidders' private signals are affiliated with that of the auctioneer. The principle also shows that an English auction, which publicly releases the buyers' private information through their actions as in the sequential scheme, generates a higher revenue than a sealed-bid second-price auction.[5] Marco Ottaviani and Andrea Prat (2001) present an alternative version of the linkage principle that is most closely related to our results. They relax the assumption of unit demand and suppose that a seller posts to a single buyer a price-quantity schedule for his good. They show that the seller should optimally commit himself to publicly revealing any information before trading as long as it is affiliated with a buyer's type and the value of the good. Their theorem also implies the optimality of sequential trading against two buyers in an

[4]Trades with other buyers take place sequentially after period 1. Sgroi (2002) shows that the optimal number of the initial buyers varies non monotonically with the size of the market.

[5]The linkage principle in its original form is limited to the standard auction framework. It may fail, for example, when the public information consists of the bids of the first-stage bidders in two-stage auctions (Maria-Angeles de-Frutos and Robert Rosenthal (1998)), when bidders are asymmetric (Vijay Krishna (2002, Ch. 8)), or when they demand multiple units (Motty Perry and Philip J. Reny (1999)).

alternative environment, and our conclusion complements this result.[6]

In marketing, the sequential sales scheme is sometimes called the "waterfall strategy," while the simultaneous sales scheme is called the "sprinkler strategy." When a firm adopts a waterfall strategy, the *lead effect* refers to the effect that consumer decisions in the first market have on those in subsequent markets. For example, Shlomo Kalish, Vijay Mahajan and Eitan Muller (1995) discuss the relative advantages of the two types of strategies by directly assuming the form of intertemporal dynamics of the lead effect. The present paper, on the other hand, can be seen as an attempt to generate the lead effect through value interdependence. Empirical research in the marketing literature also looks at the dynamic sales strategies used by multinational firms. In particular, much attention is focused on the motion picture industry where movie makers draw a detailed plan on when and how to release their products in international markets.[7] According to the aforementioned textbook by Kotler (1980), consumers are classified by their willingness to adopt a new product. Upon emphasizing that a firm's first target should be the *innovators,* those who are the most willing to adopt, he notes that the role of personal influence is stronger on those who are less willing than the more willing. Taken together, they can be interpreted as a statement on the desirable ordering of consumers based on the degree of influence they receive from other consumers' behavior. Our analysis on the optimal ordering of buyers provides one formal restatement of this theory.

The paper is organized as follows: The next section formulates a model of monopoly. In Section 3, we present an example with two buyers. Section 4 proves the optimality of a sequential scheme. In Section 5, we identify the conditions under which the optimal sequential scheme entails the increasing order of the dependence weights. Section 6 concludes with a discussion.

[6]See Section 4 for more discussion. Masaki Aoyagi (2007) also discusses the optimal information revelation in a model of a dynamic tournament with a privately informed organizer. Discussion of simultaneous versus dynamic schemes is also seen in the literature on network externalities where agents choose whether to subscribe to a network or not. See, *e.g.,* Jack Ochs and In-Uck Park (2008).

[7]For example, see Anita Elberse and Jehoshua Eliashberg (2003) and the references therein.

I. Model

A seller of a good faces the set $I = \{1, \dots, I\}$ of I buyers each of whom has private information about the valuation of the good.[8] Let s_i denote buyer i's private signal and $s = (s_1, \dots, s_I)$ be the signal profile. We assume that $s_1, \dots, s_I$ are independent and distributed over the set of real numbers. Let μ_i be the mean value of s_i. When the signal profile is $s = (s_1, \dots, s_I)$, buyer i's valuation of a single unit of the seller's good is given by

$$v_i(s) = c_{i0} + c_{ii}s_i + \sum_{j \neq i} c_{ij}(s_j - \mu_j),$$

where $c_{ij} \in \boldsymbol{R}$ are constants. In other words, the valuations are *linearly interdependent,* and buyer i places weight c_{ij} on buyer j's signal.[9] For every $i \in I$, $c_{ii} > 0$ and $c_{ij} \geq 0$ for $j \neq 0$, i. In other words, the valuation is strictly increasing in the own signal, and the buyers' preferences are aligned in the sense that any buyer having a high signal is good news for any other buyer.[10] Subtraction of the mean μ_j from s_j for every $j \neq i$ is introduced to simplify the representation of the expected valuation.[11]

We normalize the marginal cost of producing the good to zero, and assume that every buyer demands at most one unit. As discussed in the Introduction, a buyer in this model can also be interpreted as a segment of the market which has a uniform taste about the seller's good.

The seller *trades* with each buyer by posting him a price. The buyer then accepts or rejects the price and leaves the market. The price posted to each buyer and their response to it are both publicly observable.

In every period, the seller chooses target buyers and prices to offer to them as a function of past trades. Formally, denote by $I_t \subset I$ the set of buyers who are made offers in period t. An *outcome* y_t in period t is a partition (A_t, B_t) of the set I_t: A_t represents the set of buyers who have accepted their offers, and B_t represents those who have rejected their offers. For any subset J of buyers, let $Y(J)$ denote the set

[8] Note that I represents both the set and number of buyers.

[9] The additive specification of the valuation function, which is common in the auction literature, can also be interpreted as a first-order approximation to a more general function.

[10] Note that this is a natural assumption for the analysis of the probability of the same decision in Section 7. Some of our conclusions hold when $c_{ij} < 0$. See Section 8 for more discussion.

[11] Without the subtraction of μ_j, the conclusion in Section 4 holds as is while some adjustments are required for the conclusions in Section 5.

of possible outcomes from the set J of buyers. In other words, $Y(J)$ consists of all the two-way partitions of the set J including $(J, \emptyset)$ and $(\emptyset, J)$. A *history* of length t consists of the outcomes in periods $1, \ldots, t$. Let H_t denote the set of possible histories of length t, and let $H = \cup_{t=0}^{\infty} H_t$ be the set of all possible histories, where H_0 is the singleton set of the null history. Given any history $h \in H$, we denote by $I(h)$ and $U(h) = I \setminus I(h)$ the set of buyers with whom the seller has and has not, respectively, traded along h.

A *sales scheme* of the seller, denoted σ, consists of a pair of mappings $r : H \to 2^I$ and $x = (x_i)_{i \in I} : H \to \boldsymbol{R}_+^I$: r is the *target function* with $r(h)$ specifying the subset of buyers chosen for trading at history h, and x is the *pricing function* with $x_i(h)$ specifying the price offered to buyer i at h. Note in particular that the target buyers in any period can be contingent on the history. In a three-buyer model, for example, after trading with buyer 1 in period 1, the seller may choose either buyer 2 or buyer 3 in period 2 depending on whether the period 1 trade is successful or not, *etc.* It should also be noted that the specification of the price $x_i(h)$ is relevant only if i is the target buyer at h (*i.e.,* $i \in r(h)$). In order to eliminate redundancy, we require that r choose at least one buyer in every period until the list of buyers is exhausted: $r(h) \neq \emptyset$ if $U(h) \neq \emptyset$. This in particular implies that all the trading ends in or before period I. Let Σ be the set of all sales schemes. Two representative classes of sales schemes are the *simultaneous schemes* in which the seller trades with all the buyers at once (*i.e.,* $r(h) = I$ for $h \in H_0$), and the *sequential schemes* in which he trades one by one with each buyer (*i.e.,* $r(h) = \{i\}$ for some $i \in U(h)$ for each $h \in H_{t-1}$ and $t = 1, \ldots, I$).

Given a sales scheme σ, let P^σ denote the joint probability distribution of the signal profile s and the history h induced by σ. Let E^σ be the expectation with respect to the distribution P^σ. We use P without the superscript to denote the marginal distribution of s that does not depend on the sales scheme, and E to denote the corresponding expectation. For any history $h \in H$, let

$$V_i^\sigma(s_i \mid h) = E^\sigma[v_i(s_i, \tilde{s}_{-i}) \mid h]$$

be the expected valuation of buyer i with signal s_i given history h. By assumption, it can be explicitly written as

$$V_i^\sigma(s_i \mid h) = c_{i0} + c_{ii} s_i + \sum_{j \in I(h)} c_{ij}\, E^\sigma[\tilde{s}_j - \mu_j \mid h].$$

Note that the summation above is over the set $I(h)$ of buyers who have already traded along h since any other term involves the unconditional expectation of the private signal and hence cancels out. Buyer i with signal s_i accepts the seller's offer x_i at history h if and only if the expected value of the good conditional on h is greater than or equal to x_i: $V_i(s_i \mid h) \geq x_i$. The seller's expected revenue under the sales scheme σ, denoted by $\pi(\sigma)$, is simply the sum of expected payments from the I buyers.

II. Example

In this section, we illustrate the problem in a simple two-buyer example as follows: The buyers' private signals s_1 and s_2 both have the uniform distribution over the unit interval $[0,1]$ with the means $\mu_1 = \mu_2 = 1/2$. Their valuation functions are given by

$$v_1(s_1, s_2) = s_1 + c_1\Big(s_2 - \frac{1}{2}\Big) \quad \text{and} \quad v_1(s_1, s_2) = s_2 + c_2\Big(s_1 - \frac{1}{2}\Big),$$

where $0 < c_1 \leq c_2 < 2$. Note that buyer i's *ex ante* expected valuation of the good equals $V_i(s_i \mid h_0) = s_i$.

When the seller uses the simultaneous sales scheme, he will choose the price offers x_1 and x_2 so as to maximize $x_1 P(V_1(\tilde{s}_1 \mid h_0) \geq x_1)$ and $x_2 P(V_2(\tilde{s}_2 \mid h_0) \geq x_2)$, respectively. Hence, the revenue maximizing prices equal $x_1 = x_2 = 1/2$ and the seller's expected payoff equals

$$\pi^0 = \frac{1}{4} \times 2 = \frac{1}{2}.$$

On the other hand, when the seller uses the sequential sales scheme that trades with buyers 1 and 2 in this order, he needs to solve a two-step optimization problem. Consider first the problem in period 2 given the first period offer $x_1 \in [0,1]$. Let $h_1 = 1$ be the history denoting buyer 1's acceptance, and $h_1 = 0$ be the history denoting his rejection. Depending on h_1, buyer 2's valuation function is either

$$V_2(s_2 \mid 1) = s_2 + c_2 E\Big[\tilde{s}_1 - \frac{1}{2} \mid \tilde{s}_1 \geq x_1\Big] = s_2 + c_2\,\frac{x_1}{2},$$

or

$$V_2(s_2 \mid 0) = s_2 + c_2 E\Big[\tilde{s}_1 - \frac{1}{2} \mid \tilde{s}_1 < x_1\Big] = s_2 - c_2\,\frac{1 - x_1}{2}.$$

The seller also has two prices to consider in period 2: $x_2(1)$ is the price offer to buyer 2 when buyer 1 accepted in period 1, and $x_2(0)$ is the offer to buyer 2 when buyer 1 rejected. The period 2 price offers hence solve for $h_1 = 0$ and 1,

$$x_2(h_1) \in \arg\max_{x_2} x_2 \, P\Big(V_2(\tilde{s}_2 \mid h_1) \geq x_2\Big).$$

Suppose for the moment that the seller chooses

$$x_2(1) = \frac{1}{2} + c_2 \, \frac{x_1}{2}, \quad \text{and} \quad x_2(0) = \frac{1}{2} - c_2 \, \frac{1 - x_1}{2}.$$

These prices are obtained by adding to the optimal price $\frac{1}{2}$ under the simultaneous scheme the change in buyer 2's valuation $V_2(s_2 \mid h_1) - V_2(s_2 \mid h_0)$ ($h_1 = 0$, 1) from the ex ante level. While these prices may be suboptimal, they yield the seller the expected period 2 revenue of

$$\begin{aligned}
&P(\tilde{s}_1 \geq x_1) \, x_2(1) \, P\Big(V_2(\tilde{s}_2 \mid 1) \geq x_2(1)\Big) + P(\tilde{s}_1 < x_1) \, x_2(0) \, P\Big(V_2(\tilde{s}_2 \mid 0) \geq x_2(0)\Big) \\
&= P(\tilde{s}_1 \geq x_1) \left(\frac{1}{2} + c_2 \, \frac{x_1}{2}\right) P\Big(\tilde{s}_2 \geq \frac{1}{2}\Big) + P(\tilde{s}_1 < x_1) \left(\frac{1}{2} - c_2 \, \frac{1 - x_1}{2}\right) P\Big(\tilde{s}_2 \geq \frac{1}{2}\Big) \\
&= \frac{1}{2} \, P\Big(\tilde{s}_2 \geq \frac{1}{2}\Big),
\end{aligned}$$

which equals his expected revenue from buyer 2 in the simultaneous scheme. Hence, the sequential scheme with $x_1 = 1/2$, and $x_2(0)$ and $x_2(0)$ defined as above yields the same expected revenue as the optimal simultaneous scheme. On the other hand, the optimal prices given x_1 are computed as

$$x_2(1) = \frac{1}{2}\Big(1 + \frac{c_2}{2} \, x_1\Big), \quad \text{and} \quad x_2(0) = \frac{1}{2}\Big(1 - \frac{c_2}{2} \, (1 - x_1)\Big).$$

They yield the expected period 2 revenues of

$$\pi_2(x_1 \mid 1) = \frac{1}{4} \left(1 + \frac{c_2}{2} \, x_1\right)^2, \quad \text{and} \quad \pi_2(x_1 \mid 0) = \frac{1}{4} \left(1 - \frac{c_2}{2} \, (1 - x_1)\right)^2 \tag{1}$$

after $h_1 = 1$ and $h_1 = 0$, respectively. The seller's period 1 maximization problem can be written as:

$$\max_{x_1} P(\tilde{s}_1 \geq x_1) \Big\{x_1 + \pi_2(x_1 \mid 1)\Big\} + P(\tilde{s}_1 < x_1) \, \pi_2(x_1 \mid 0).$$

Solve this to get

$$x_1 = \frac{1}{2}.$$

The seller's maximized expected payoff under the sequential scheme with buyer 1 first hence equals

$$\pi^{12} = \frac{1}{2} + \frac{c_2^2}{64}.$$

Likewise, when the seller uses the sequential scheme with the order of buyers 1 and 2 reversed, his maximized expected payoff is given by

$$\pi^{21} = \frac{1}{2} + \frac{c_1^2}{64}.$$

Given our assumptions on c_1 and c_2, we hence have the following ordering:

$$\pi^0 < \pi^{21} \leq \pi^{12}. \tag{2}$$

The inequalities in (2) show that each sequential scheme is better than the simultaneous scheme, and that the sequential scheme with an increasing order of the weights is better than the sequential scheme with the reverse order. The superiority of the simultaneous schemes in this two-buyer example can be understood in the context of public revelation of payoff relevant information. The sequential schemes supply both the seller and the second buyer with more information about the value of the good, and contingent adjustment in the price offer to the second buyer allows the seller to raise a higher revenue as seen above. As for the ranking between the two sequential schemes, note from (1) that increasing c_2 increases the seller's period 2 revenue when he has a success in period 1, but decreases it if he has a failure in period 1. For relevant values of x_1, the magnitude of the first impact is larger than that of the second because of the sign of the constant term. This suggests that having a more dependent buyer in period 2 is good for the seller. These observations are generalized in the following sections.

While it is common in the theory of mechanism design to resort to the revelation principle and study the properties of direct mechanisms, the focus of the present paper is on price posting mechanisms.[12] Price posting is not only an important sales mechanism in its own right, but also is more plausible than a direct mechanism in some ways. Most importantly, in a sales situation like the one considered in this paper, we rarely observe the practice of using private signals solicited from buyers to force an allocation on them as is done by a direct mechanism.[13]

[12] In the present setup, a direct scheme would first solicit private signals from all the buyers, and then choose the allocation of the good and monetary transfer to each buyer as a function of the reported signal profile.

[13] At more technical levels, the standard analysis of a direct mechanism has the following problems: First, it does not exclude the possibility that a buyer's payment is positive even when he does not obtain the good, or that it is contingent on another buyer's report. The positive payment is difficult to justify with no competition for the good, while the contingent price leads to the seller's credibility problem. Second, it imposes individual rationality at the interim stage. This is insufficient when a buyer can leave the market after seeing the price offer.

III. Optimality of Sequential Sales Schemes

In the example of the preceding section, the revenue from each alternative scheme can be computed explicitly. However, technical complexity makes such computation simply infeasible in a general problem. For this reason, we take a different approach in what follows and examine how a local change in a given scheme affects the revenue. In this section, we show that the seller's expected payoff is maximized when he employs a sequential scheme.

Theorem 1. *The seller's expected revenue is maximized when he employs a sequential scheme: There exists a sequential sales scheme σ^* such that $\pi(\sigma^*) = \max_{\sigma \in \Sigma} \pi(\sigma)$.*

Proof. See the Appendix.

The proof of the theorem shows that given any non sequential scheme σ, there exists a sequential scheme that performs at least as well as σ. Suppose for simplicity that σ induces some history $\underline{h} \in H_{n-1}$ $(n \geq 1)$ at which it trades with two buyers i and m. Let $x_i \equiv x_i(\underline{h})$ denote the price offer to buyer i under the original scheme. Consider the following alternative scheme $\sigma^* = (r^*, x^*)$: In period n at history $\underline{h}$, σ^* trades only with buyer m by offering the same price as under σ. Let $(\underline{h}, 1) \in H_n$ be the history under σ^* denoting m's acceptance at $\underline{h}$, and $(\underline{h}, 0) \in H_n$ be the history denoting his rejection at $\underline{h}$. Let

$$\begin{aligned} \kappa_m &= E[\tilde{s}_m - \mu_m \mid V_m^{\sigma}(\tilde{s}_m \mid \underline{h}) \geq x_m(\underline{h})], \\ \lambda_m &= E[\tilde{s}_m - \mu_m \mid V_m^{\sigma}(\tilde{s}_m \mid \underline{h}) < x_m(\underline{h})] \end{aligned} \tag{3}$$

be the expected values of m's private signal (minus the unconditional mean μ_m) conditional on $(\underline{h}, 1)$ and $(\underline{h}, 0)$, respectively. The change in buyer i's valuation in period n equals either $c_{im}\,\kappa_m$ or $c_{im}\,\lambda_m$ depending on m's decision. The contingent offer to i in period $n+1$ is set equal to $x_i + c_{im}\,\kappa_m$ at $(\underline{h}, 1)$, and $x_i + c_{im}\,\lambda_m$ at $(\underline{h}, 0)$. Since the adjustment is exactly in line with the change in i's valuation, the probability that this offer is accepted at $(\underline{h}, 1)$ or $(\underline{h}, 1)$ is the same as the probability that x_i is accepted at $\underline{h}$ under the original scheme. Formally, if we denote by $z_i^*(\underline{h}, 1)$ the probability that i accepts at $(\underline{h}, 1)$ under σ^*, then

$$\begin{aligned} z_i^*(\underline{h}, 1) &= P\Big(V_i^{\sigma^*}(\tilde{s}_i \mid \underline{h}, 1) \geq x_i + c_{im}\,\kappa_m\Big) \\ &= P\Big(V_i^{\sigma}(\tilde{s}_i \mid \underline{h}) \geq x_i\Big) \\ &= z_i(\underline{h}). \end{aligned}$$

Likewise, the probability $z_i^*(\underline{h}, 0)$ that i accepts at $(\underline{h}, 0)$ under σ^* equals $z_i(\underline{h})$. It then follows that the seller's expected revenue from buyer i under σ^* conditional on $\underline{h}$ is computed as

$$\begin{aligned}
& z_m(\underline{h})\, z_i(\underline{h}) \{x_i + c_{im}\, \kappa_m\} + (1 - z_m(\underline{h}))\, z_i(\underline{h}) \{x_i + c_{im}\, \lambda_m\} \\
& = z_m(\underline{h}) \left[x_i + c_{im} \left\{ z_m(\underline{h})\, \kappa_m + (1 - z_m(\underline{h}))\, \lambda_m \right\} \right] \\
& = z_i(\underline{h}) x_i,
\end{aligned}$$

where the second equality follows since $z_m(\underline{h})\, \kappa_m + (1 - z_m(\underline{h}))\, \lambda_m = 0$ by the definitions of κ_m and λ_m. Note that $z_i(\underline{h}) x_i = z_i(\underline{h}) x_i(\underline{h})$ is just the expected revenue from buyer i under the original scheme. As shown in Lemma A.1 in the Appendix, the valuation function of any subsequent buyer (that comes after i and m) is unchanged from that under σ given that the offers to i and m under σ^* are accepted with the same probability as those under σ. Therefore, by making the same offer to each subsequent buyer as σ, σ^* raises exactly the same revenue from them. In other words, we can locally "expand" σ at $\underline{h}$ so as not to affect the seller's revenue. The argument here generalizes to the case where σ chooses more than two buyers at $\underline{h}$, and hence the optimality of a sequential scheme is obtained.

As mentioned in the Introduction, Ottaviani and Prat (2001) prove the linkage principle for the monopoly problem when the monopolist can publicly reveal information which is affiliated with the value of the good as well as the buyer type. Their theorem specifically implies that a sequential scheme dominates a simultaneous scheme since the former provides the buyers with more information. However, it cannot be applied to our framework for the following reasons. First, the analysis of Ottaviani and Prat (2001) builds on the incentive compatibility conditions associated with finite buyer types. It is hence not clear how we can generalize their argument to the continuous type distribution. Second, and more important, their result cannot be used to rank various intermediate schemes that may arise when there are three or more buyers. To see this point, suppose that we want to compare the performance of the following two schemes against three buyers: In the first scheme, the seller trades with buyers 1 and 2 in stage 1 and then with buyer 3 in stage 2. In the second sequential scheme, the seller trades with buyer 1 in stage 1, buyer 2 in stage 2, and buyer 3 in stage 3. The theorem of Ottaviani and Prat (2001) shows that the seller's revenue from buyer 2 is higher in the second scheme than in the first scheme since buyer 2 is provided with more information in the second scheme. However, it is not clear if the seller's revenue from buyer 3 is likewise increased since he observes the decisions of the other two buyers in both schemes. On the other hand, we show that a local expansion of a non sequential scheme can be done in such a manner that the expected revenue from buyer 3 remains the same.

IV. Optimal Buyer Ordering

We now focus on the optimal ordering of buyers in a sequential scheme when they differ only in the weights they place on the others' signals. Suppose specifically that each buyer's private signal has a common distribution F, and that his valuation $v_i(s)$ given the signal profile $s = (s_1, \ldots, s_I)$ equals

$$v_i(s) = c_0 + s_i + c_i \sum_{j \neq i} (s_j - \mu),$$

where μ is the common mean of s_j.[14] Note that c_i, which is the only source of difference across buyers, measures the degree of dependence of buyer i's valuation on others' information.

Assume now that the common distribution F of each s_i is the uniform distribution over $[\underline{s}, \bar{s}]$ ($\Delta = \bar{s} - \underline{s}$). Under this assumption that the theorem below provides sufficient conditions for the optimal scheme to trade in the increasing order of the dependence weights. It essentially requires that the dependence weights be not very large, similar in size, and that the support of the distribution be sufficiently large compared with $|\underline{s} + c_0|$:[15]

Theorem 2. *Suppose that each s_i has the uniform distribution over $[\underline{s}, \bar{s}]$ ($\Delta = \bar{s} - \underline{s}$), and that $c_1, \ldots, c_I$ satisfy $c_1 < \cdots < c_I$, and*

$$c_I < \frac{2}{I-1} - \beta, \quad \textit{and} \quad \frac{c_I}{c_1} < 1 + \frac{\beta^2}{16} \quad \textit{for some } \beta \in (0, \frac{2}{I-1}). \tag{4}$$

Then there exists $\delta \in (0,1)$ such that if $\frac{|\underline{s}+c_0|}{\Delta} < \delta$, then the optimal sequential scheme trades with buyer i in period i ($i = 1, \ldots, I$) regardless of the outcome of transactions in periods $1, \ldots, i-1$.

Proof. See the Appendix.

[14]That is, we set $c_{ii} = 1$, $c_{ij} = c_i$ for $j \neq i$, 0, and $c_{i0} = c_0$ in the general formulation of Section 2.

[15]Note from Theorem 2 that the total weight placed on the other buyers' signals ($= (I-1)\, c_i$) need not be very small. Note also that the condition on the support is not needed if $\underline{s} + c_0 = 0$.

The proof of this result involves formulating the seller's maximization problem when he controls contingent prices against a fixed sequence of buyers, and identifying a sufficient condition in terms of the contingent pricing function under which the optimal scheme should trade in the increasing order of weights. We then solve for the optimal contingent pricing function for the uniform distribution, and verify that the above sufficient condition is implied by the conditions of Theorem 2. The restriction to the uniform distribution stems from the difficulty of obtaining an analytical expression for the optimal contingent pricing function, which is a solution to a dynamic programming problem. To complement Theorem 2, we numerically solved for the optimal sequential scheme in an environment not covered above. Specifically, we checked the environments where (i) s_i has a uniform distribution but the dependence weights do not satisfy (4), and (ii) s_i has a (truncated) exponential distribution, and found in both cases that the optimal sequential scheme trades in the increasing order of the weights.[16]

One implication of Theorem 2 concerns the effect of buyer heterogeneity on the probability that every buyer makes the same decision under the optimal scheme. When the seller uses the optimal sequential scheme in the framework of the above theorem, we observe that the buyers are more likely to choose the same action as others when they are heterogeneous than when they are homogeneous.[17] Such a result is of interest in view of the consumption fads in some markets, and also of the significant attention focused on such behavior in the literature on social learning. Formally, given the original environment with weights $c_1, \dots, c_I$, consider an alternative environment where the buyers are ex ante identical and their dependence weights $c'_1, \dots, c'_I$ all equal the average $\bar{c} \equiv \frac{1}{I} \sum_{i=1}^{I} c_i$ of the original environment. Let σ and σ' be the optimal sequential schemes under $(c_1, \dots, c_I)$ and $(c'_1, \dots, c'_I)$, respectively. The following proposition states that the probability of the same decision under the optimal scheme is higher with heterogeneous buyers. For technical reasons, we suppose that weights $c_1, \dots, c_I$ are small, and consider the limit as $\varepsilon \to 0$ in

$$c_i = \varepsilon \hat{c}_i \text{ for every } i \in I, \tag{5}$$

where $\hat{c}_1, \dots, \hat{c}_I$ are constants which satisfy

$$\hat{c}_1 < \cdots < \hat{c}_I, \quad \text{and} \quad \frac{\hat{c}_I}{\hat{c}_1} < 1 + \frac{1}{4(I-1)^2}. \tag{6}$$

[16] The details can be found in the online supplement to this paper.

[17] The supplement to this paper presents another related implication of Theorem 2.

The conditions on $c_1, \ldots, c_I$ of Theorem 2 then hold for a small ε.

Proposition 1. *Suppose that every s_i has the uniform distribution, that (5) and (6) hold, and that $\underline{s} + c_0 = 0$.*[18] *Then for a sufficiently small $\varepsilon > 0$, the probabilities that the buyers all accept and that they all reject are higher under σ than under σ'.*

V. Discussions

As emphasized in the Introduction, careful choice of the timing of trades is at the core of the design of a good sales strategy. Our conclusions shed light on the dynamic sales strategies used in reality as studied extensively in the marketing literature. We below provide discussions on the key assumptions of our model.

– Independence of private signals and the linearity of value functions. As seen in Section 3, these assumptions imply that a buyer's valuation changes by a publicly known amount in response to the outcome of each preceding transaction. This property no longer holds when the signals are correlated or when the valuation functions are multiplicative. In both cases, the impact of each transaction on a buyer's valuation varies with his own signal.[19] The computation of the seller's revenue in these cases must sort through a layer of private expectations. While our assumptions are restrictive, they lead to a considerable simplification of the analysis. At the same time, we suspect that slight perturbations of these assumptions would not change the derived properties of the optimal scheme.

– No cost of producing the good for the seller. When the production cost is positive and must be incurred before each buyer's decision, we would need to consider the possibility of exit by the seller when he finds the buyers' valuations to be too low to justify further production. For example, in a model of monopoly pricing against the sequence of buyers, Bose *et al.* (2007) identify the range of beliefs that such exit takes place. Another implicit assumption of the present paper is that the seller cannot gain by limiting the supply of his good. That is, the seller cannot improve

[18]The last condition is assumed for simplicity.

[19]With correlated signals, the seller also faces the learning problem. For example, if the private signals indicate the underlying common value of the good, it would be in the interest of the seller to engage in experimental pricing against the initial set of buyers. As seen in Bose *et al.* (2006, 2007), these issues significantly complicate the analysis.

his revenue by, for example, supplying only five units to ten buyers.[20]

− Sign of the dependence weights. We assume that the buyers' preferences are aligned in the sense that the weight c_{ij} that buyer i's valuation places on j's signal is non negative. This assumption is not required for some of our conclusions. Specifically, the optimality of a sequential scheme is independent of the signs of the weights. When the weights are all negative, the same conclusion on the optimal ordering of buyers holds under the uniform distribution of the signals.

− Modeling transactions as a take-it-or-leave-it offer. We assume that the buyers make a binary decision at the seller's offer and leave the market once they reject it. One possible interpretation of this assumption is that acceptance/rejection is not an action of a single buyer, but is rather a crude summary of the aggregate behavior of consumers in the market segment represented by the buyer. In other words, while the interaction between the seller and consumers in each market may be much more complex than described by a simple take-it-or-leave-it offer, the consumers outside the market have limited capabilities and can perceive the outcome only as a simple success or failure. One obvious way to enrich the model at hand is to give the seller multiple chances to approach buyers with different price offers. To make such a model interesting, we need both the seller and buyers to discount future as in the case of durable good monopoly, and the conclusion would critically depend on the level of discounting. With discounting, of course, the optimality of sequential trading holds only with qualifications.

As discussed in the Introduction, an alternative interpretation of the present model is through the seller's information revelation policy. A more direct model of information revelation would be obtained if we assume that each buyer observes only the outcome of his own transaction. In such a setup, the seller's information policy specifies which past outcomes to reveal to each subsequent buyer as a function of history. The seller's credibility becomes a critical issue in such a model.

[20]One way to justify this is to assume that selling at the minimum price to every buyer is more profitable than selling at the maximum price to any proper subset of them.

Appendix

Preliminaries

We begin with a lemma (Lemma A.1) which is used in the proofs of Theorems 1 and 2. Consider a pair of sales schemes σ and σ', and suppose that a pair of histories h and h' are induced by σ and σ', respectively. Suppose that along these histories, the seller has traded with the same set of buyers with exactly the same outcomes. That is, the set of buyers who have accepted the seller's offers along h is the same as that along h' ($A(h) = A(h')$), and also the set of the buyers who have rejected the offers along h is the same as that along h' ($B(h) = B(h')$). The following lemma states that if, for every one of those buyers, the probability that he accepts the offer is the same under both schemes, then so are the valuation functions of subsequent buyers conditional on h and h'. It is based on the following simple logic: No matter what the history up to buyer j is, if the probability that j accepts his offer under one scheme is the same as that under another scheme, then the inference drawn about j's private signal when he accepts (resp. rejects) the offer in the first scheme is the same as that when he accepts (resp. rejects) the offer in the second scheme. Formally, given any sales scheme $\sigma = (r, x) \in \Sigma$ and any history $h \in H$, let

$$z_i^\sigma(h) = P\Big(V_i^\sigma(\tilde{s}_i \mid h) \geq x_i(h)\Big)$$

be the probability that buyer i accepts the seller's offer $x_i(h)$ given his valuation conditional on history $h \in H$.

Lemma A.1. *Let $\sigma = (r, x)$ and $\sigma' = (r', x')$ be any sales schemes and h and h' be any histories induced by σ and σ', respectively, with the same set of buyers along them and the same outcomes (i.e., $A(h) = A(h')$ and $B(h) = B(h')$). For any buyer $j \in J \equiv I(h) = I(h')$, let h_j and h'_j denote the truncations of h and h', respectively, at which the seller trades with j: $j \in r(h_j) \cap r'(h'_j)$. If $z_j^\sigma(h_j) = z_j^{\sigma'}(h'_j)$ for every $j \in J$, then for any remaining buyer $i \notin J$,*

$$V_i^\sigma(s_i \mid h) = V_i^{\sigma'}(s_i \mid h') \text{ for every } s_i.$$

Proof of Lemma A.1 Take any $j \in A(h) = A(h')$. Write

$$w_j^\sigma(h_j) = c_{j0} + \sum_{k \in I(h_j)} c_{jk} \, E^\sigma[\tilde{s}_k - \mu_k \mid h_j]$$

for the part of j's valuation under σ that is determined by the history h_j. Likewise, define $w_j^{\sigma'}(h'_j)$ to be the part of j's valuation under σ' that is determined by the history h'_j. Since j's valuation is given by $V_j^{\sigma}(s_j \mid h_j) = c_{jj}s_j + w_j^{\sigma}(h_j)$ and $V_j^{\sigma'}(s_j \mid h'_j) = c_{jj}s_j + w_j^{\sigma'}(h'_j)$, we have by assumption,

$$z_j^{\sigma}(h_i) = P\Big(c_{jj}\tilde{s}_j \geq x_j(h_j) - w_j^{\sigma}(h_j)\Big) = P\Big(c_{jj}\tilde{s}_j \geq x'_j(h'_j) - w_j^{\sigma'}(h'_j)\Big) = z_j^{\sigma'}(h'_j).$$

Hence

$$\begin{aligned} E^{\sigma}[\tilde{s}_j - \mu_j \mid h] &= E[\tilde{s}_j - \mu_j \mid V_j^{\sigma}(\tilde{s}_j \mid h_j) \geq x_j(h_j)] \\ &= E[\tilde{s}_j - \mu_j \mid c_{jj}\tilde{s}_j \geq x_j(h_j) - w_j^{\sigma}(h_j)] \\ &= E[\tilde{s}_j - \mu_j \mid c_{jj}\tilde{s}_j \geq x'_j(h'_j) - w_j^{\sigma'}(h'_j)] \\ &= E[\tilde{s}_j - \mu_j \mid V_j^{\sigma'}(\tilde{s}_j \mid h'_j) \geq x'_j(h'_j)] \\ &= E^{\sigma'}[\tilde{s}_j - \mu_j \mid h']. \end{aligned} \tag{A.1}$$

Likewise, for any $j \in B(h) = B(h')$, we have $E^{\sigma}[\tilde{s}_j - \mu_j \mid h] = E^{\sigma'}[\tilde{s}_j - \mu_j \mid h']$. Now take any buyer $i \notin J$ who has not traded along h or h'. Since for any s_i, $V_i^{\sigma}(s_i \mid h) = c_{i0} + c_{ii}s_i + \sum_{j\in J} c_{ij}\, E^{\sigma}[\tilde{s}_j - \mu_j \mid h]$ and $V_i^{\sigma'}(s_i \mid h') = c_{i0} + c_{ii}s_i + \sum_{j\in J} c_{ij}\, E^{\sigma'}[\tilde{s}_j - \mu_j \mid h']$, we conclude from the above that $V_i^{\sigma}(s_i \mid h) = V_i^{\sigma'}(s_i \mid h')$.

Proof of Theorem 1

Fix any sales scheme σ that is not sequential. That is, σ induces a history $\underline{h} \in H_{n-1}$ ($n \geq 1$) such that $r(\underline{h}) = \{m\} \cup J$ for some $m \in I$ and $J \neq \emptyset$. In other words, according to σ, the seller trades with buyer m and at least one other buyer in period n at history $\underline{h}$. We will construct an alternative scheme σ^* that raises the same expected revenue as σ as follows: The sales scheme σ^* operates in the same way as σ does except when $\underline{h}$ arises. At history $\underline{h}$, σ^* trades only with buyer m with the same offer price $x_m = x_m(\underline{h})$ as under the original scheme. Denote the outcome $y_n \in Y(\{m\})$ in period n from buyer m under σ^* by either 0 or 1: 1 represents the outcome $(\{m\}, \emptyset)$ that buyer m accepts the seller's offer, and 0 represents the outcome $(\emptyset, \{m\})$ that he rejects it. In period $n+1$ at either $(\underline{h}, 1)$ or $(\underline{h}, 0)$, σ^* trades with the buyers in J with the offer prices adjusted according to the outcome in period n. In any subsequent period, the set of buyers and prices specified by σ^* along any history $(\underline{h}, y_n, \ldots, y_{t-1}) \in H_{t-1}$ are the same as those specified by σ along the history $(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) \in H_{t-2}$, where $y_n \cup y_{n+1} = (A_n \cup A_{n+1}, B_n \cup B_{n+1})$ is the "union" of two outcomes y_n and y_{n+1}: Those who accept under $y_n \cup y_{n+1}$

are the union of those who accept under y_n and y_{n+1}, and those who reject under $y_n \cup y_{n+1}$ are the union of those who reject under $y_n \cup y_{n+1}$. In other words, σ^* operates just as σ by assuming that the outcomes in periods n and $n+1$ came from the same period. Let κ_m and λ_m be defined as in (3). A formal description of σ^* is given as follows:

$$r^*(h) = \begin{cases} \{m\} & \text{if } h = \underline{h}, \\ J & \text{if } h = (\underline{h}, 1) \text{ or } (\underline{h}, 0), \\ r(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \dots, y_{t-1}) & \\ & \text{if } h = (\underline{h}, y_n, \dots, y_{t-1}) \text{ for some} \\ & y_n, \dots, y_{t-1} \ (t \geq n+2), \\ r(h) & \text{otherwise.} \end{cases} \tag{A.2}$$

and for any $i \in I$,

$$x_i^*(h) = \begin{cases} x_i(\underline{h}) + c_{im}\kappa_m & \text{if } h = (\underline{h}, 1) \\ x_i(\underline{h}) + c_{im}\lambda_m & \text{if } h = (\underline{h}, 0) \\ x_i(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \dots, y_{t-1}) & \\ & \text{if } h = (\underline{h}, y_n, \dots, y_{t-1}) \text{ for some} \\ & y_n, \dots, y_{t-1} \ (t \geq n+2), \\ x_i(h) & \text{otherwise.} \end{cases} \tag{A.3}$$

In what follows, we will show that σ^* yields the same expected revenue as σ. Since σ is an arbitrary non sequential scheme, repeated application of this argument shows that for any scheme σ that is not sequential, there exists a sequential scheme that yields the same expected payoff as σ. The desired conclusion would then follow.

For simplicity, denote

$$V_i(s_i \mid h) = V_i^{\sigma}(s_i \mid h), \quad V_i^*(s_i \mid h) = V_i^{\sigma^*}(s_i \mid h).$$

For buyer $i \notin I(\underline{h})$, let also

$$w_i = c_{i0} + \sum_{j \in I_{n-1}} c_{ij} E^{\sigma}[\tilde{s}_j - \mu_j \mid \underline{h}].$$

be the component of i's valuation that is already determined along $\underline{h}$. Note that

$$V_i(s_i \mid \underline{h}) = V_i^*(s_i \mid \underline{h}) = c_{ii}s_i + w_i, \tag{A.4}$$

and for any outcome $y_n \in Y(\{m\}) = \{0,1\}$ from buyer m in period n,

$$V_i^*(s_i \mid \underline{h}, y_n) = \begin{cases} c_{ii}s_i + w_i + c_{im}\kappa_m & \text{if } y_n = 1, \\ c_{ii}s_i + w_i + c_{im}\lambda_m & \text{if } y_n = 0. \end{cases} \tag{A.5}$$

It hence follows from (A.3) that for any $i \in J$,

$$\begin{aligned} z_i^*(\underline{h}, y_n) &= P\Big(V_i^*(\tilde{s}_i \mid \underline{h}, y_n) \geq x_i^*(\underline{h}, y_n)\Big) \\ &= P\Big(c_{ii}\tilde{s}_i + w_i \geq x_i(\underline{h})\Big) \\ &= P\Big(V_i(\tilde{s}_i \mid \underline{h}) \geq x_i(\underline{h})\Big) \\ &= z_i(\underline{h}). \end{aligned} \tag{A.6}$$

It then follows from Lemma A.1 that

$$V_i^*(s_i \mid \underline{h}, y_n, y_{n+1}) = V_i(s_i \mid \underline{h}, y_n \cup y_{n+1}).$$

For any $t \geq n+2$ and any sequence of outcomes $y_n, \ldots, y_{t-1}$ in periods $n, \ldots, t-1$ under σ^*, we will show that a buyer's valuation function $V_i^*(\cdot \mid \underline{h}, y_n, \ldots, y_{t-1})$ in period t induced by σ^* is the same as the valuation function $V_i(\cdot \mid \underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1})$ in period $t-1$ induced by σ. As an induction hypothesis, suppose that

$$V_i^*(s_i \mid \underline{h}, y_n, \ldots, y_{t-1}) = V_i(s_i \mid \underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1})$$

for some $t \geq n+2$. Since

$$x_i^*(\underline{h}, y_n, \ldots, y_{t-1}) = x_i(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1})$$

by definition, we have

$$z_i^*(\underline{h}, y_n, \ldots, y_{t-1}) = z_i(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}).$$

Hence, Lemma A.1 implies that

$$V_i^*(s_i \mid \underline{h}, y_n, \ldots, y_t) = V_i(s_i \mid \underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_t).$$

For any $h \in H_{t-1}$, let $\pi_t(h)$ denote the seller's expected revenue in periods $t, \ldots, I$ at history h when he employs the sales scheme σ. Define $\pi_t^*(h)$ similarly for σ^*.

Given the equality of the valuation functions induced by the two schemes as seen above, we have

$$\pi^*_{n+2}(\underline{h}, y_n, y_{n+1}) = \pi_{n+1}(\underline{h}, y_n \cup y_{n+1})$$

for any sequence of outcomes (y_n, y_{n+1}) in periods n and $n+1$ under σ^*. On the other hand,

$$\begin{aligned}\pi^*_n(\underline{h}) = &\sum_{y_n \in Y(\{m\})} \sum_{y_{n+1} \in Y(J)} P^{\sigma^*}(y_n \mid \underline{h})\, P^{\sigma^*}(y_{n+1} \mid \underline{h}, y_n) \\ &\cdot \Big\{ \sum_{i \in A_n} x^*_i(\underline{h}) + \sum_{i \in A_{n+1}} x^*_i(\underline{h}, y_n) + \pi^*_{n+2}(\underline{h}, y_n, y_{n+1}) \Big\}.\end{aligned} \tag{A.7}$$

Likewise, the expected revenue in period n under σ conditional on $\underline{h}$ can be expressed using $Y(\{m\})$ and $Y(J)$ as:

$$\begin{aligned}\pi_n(\underline{h}) = &\sum_{y_n \in Y(\{m\})} \sum_{y_{n+1} \in Y(J)} P(y_n \mid \underline{h})\, P(y_{n+1} \mid \underline{h}) \\ &\cdot \Big\{ \sum_{i \in A_n} x_i(\underline{h}) + \sum_{i \in A_{n+1}} x_i(\underline{h}) + \pi_{n+1}(\underline{h}, y_n \cup y_{n+1}) \Big\}.\end{aligned} \tag{A.8}$$

Since σ and σ^* are identical up to and including period $n-1$, we have for $y_n \in Y(\{m\}) = \{0, 1\}$,

$$P^{\sigma^*}(y_n \mid \underline{h}) = P^{\sigma}(y_n \mid \underline{h}). \tag{A.9}$$

By (A.6), we also have for any $y_{n+1} \in Y(J)$,

$$\begin{aligned}P^{\sigma^*}(y_{n+1} \mid \underline{h}, y_n) &= \prod_{i \in A_{n+1}} z^*_i(\underline{h}, y_n) \prod_{i \in B_{n+1}} (1 - z^*_i(\underline{h}, y_n)) \\ &= \prod_{i \in A_{n+1}} z_i(\underline{h}) \prod_{i \in B_{n+1}} (1 - z_i(\underline{h})) \\ &= P^{\sigma}(y_{n+1} \mid \underline{h}).\end{aligned} \tag{A.10}$$

Using (A.8), (A.10) and (A.9), and substituting the definitions of $x^*_i(\underline{h})$ and $x^*_i(\underline{h}, y_n)$, we can rewrite (A.7) as:

$$\begin{aligned}\pi^*_n(\underline{h}) = \pi_n(\underline{h}) + &\sum_{y_{n+1} \in Y(J)} P^{\sigma}(y_{n+1} \mid \underline{h}) \\ \cdot \sum_{i \in A_{n+1}} \Bigg[&\sum_{y_n \in Y(\{m\})} P^{\sigma}(y_n \mid \underline{h}) \Big\{ \sum_{j \in A_n} c_{ij} \kappa_j(\underline{h}) + \sum_{j \in B_n} c_{ij} \lambda_j(\underline{h}) \Big\} \Bigg],\end{aligned} \tag{A.11}$$

where the order of the summations in the second term is reversed since their ranges are independent of each other. Since $Y(\{m\}) = \{(\emptyset, \{m\}), (\{m\}, \emptyset)\}$, the quantity in the square brackets on the right-hand side of (A.11) equals

$$\begin{aligned} &\sum_{y_n \in Y(\{m\})} P(y_n \mid \underline{h}) \Big\{ \sum_{j \in A_n} c_{ij} \kappa_j(\underline{h}) + \sum_{j \in B_n} c_{ij} \lambda_j(\underline{h}) \Big\} \\ &= c_{im} \Big\{ z_m(\underline{h})\, \kappa_m + (1 - z_m(\underline{h}))\, \lambda_m \Big\} \\ &= 0. \end{aligned} \tag{A.12}$$

This completes the proof of the theorem.

Proof of Theorem 2

Assume that the distribution F is strictly increasing and has bounded support $[\underline{s}, \bar{s}]$.[21] Given a sequential sales scheme $\sigma = (r, x)$, we redefine $r(h)$ to be the buyer (an element of I) that r chooses at history h. Given any history $h \in H_{n-1}$ ($n \leq I - 1$), the sales scheme $\sigma = (r, x)$ has a *fixed order after* h if the buyers chosen in periods $n + 1, \ldots, I$ are independent of the outcomes in periods $n, \ldots, I - 1$, *i.e.,* there exist $\rho_{n+1}, \ldots, \rho_I \in I \setminus I(h)$ such that for any sequence of outcomes $y_n, \ldots, y_{t-1}$ in periods $n, \ldots, t$,

$$r(h, y_n, \ldots, y_{t-1}) = \rho_t. \tag{A.13}$$

In other words, σ has a fixed order after h if the target buyers in all future periods are known at h.[22] Note that for any history h of length $I-2$, every sequential scheme σ has a fixed order after h since only one buyer remains in period I no matter what happens with buyer $r(h)$ in period $I-1$. Given a permutation $\rho = (\rho_1, \ldots, \rho_I)$ over I, σ has a *fixed order* ρ if it has a fixed order after the null history, and trades with buyers $\rho_1, \ldots, \rho_I$ in this order.

The proof of the theorem is outlined as follows: In Step 1, we formulate a sub-optimization problem called a sequential pricing problem. This is a problem of revenue maximization by the seller when he controls price offers against a fixed sequence of subset $J \subset I$ of buyers. Step 2 presents Lemma A.2, which states that if the maximized revenue against any subset of buyers is improved when we bring the buyer with the smallest weight to the top, then the optimal sales scheme has a

[21] Although F is later assumed to be a uniform distribution, the discussions in Steps 1-3 below do not depend on this assumption.

[22] Note that the target buyer $r(h)$ in period n is known at h whether σ has a fixed order or not.

fixed, increasing order of weights c_i's. In Step 3, we provide a sufficient condition for Lemma A.2 in terms of the solution to the sequential pricing problem. Step 4 presents an explicit solution to the sequential pricing problem when each signal s_i has the uniform distribution. Finally in Step 5, we show that the conditions of Theorem 2 do in fact imply that the solution for the uniform distribution satisfies the sufficient condition identified in Step 3.

Step 1: Sequential Pricing Problem

We begin the analysis of optimal sequential schemes by the discussion of the optimal contingent pricing against a fixed sequence of a subset of buyers. The interpretation is that those buyers are at the tail of the sequence in the original scheme. Let a sales scheme $\sigma \in \Sigma$ be given. For any history $h \in H_{t-1}$, let

$$\alpha^\sigma(h) = \sum_{j \in I(h)} E^\sigma[\tilde{s}_j - \mu \mid h]$$

be the sum of the expected values of private signals (minus μ) conditional on history h. Using $\alpha^\sigma(h)$, we can express buyer i's valuation function conditional on history h as

$$V_i^\sigma(s_i \mid h) = c_0 + s_i + c_i \alpha^\sigma(h). \tag{A.14}$$

$\alpha^\sigma(h)$ is referred to as the *state* at h since it completely determines a buyer's valuation at h. In what follows, we write $V_i(s_i \mid \alpha)$ for buyer i's valuation in state α:

$$V_i(s_i \mid \alpha) = c_0 + s_i + c_i \alpha.$$

We also redefine the pricing function x_i as a function of α, and let $z_i(\alpha)$ denote the probability that the offer $x_i(\alpha)$ is accepted by buyer i in state α. Since

$$z_i = P(c_0 + \tilde{s}_i + c_i\alpha \geq x_i) = 1 - F(x_i - c_0 - c_i\alpha),$$

x_i can be expressed in terms of z_i as

$$x_i = F^{-1}(1 - z_i) + c_i\alpha + c_0 \tag{A.15}$$

in state α. Buyer i accepts this offer if and only if $V_i(s_i \mid \alpha) = c_0 + c_i\alpha + s_i \geq x_i$, or equivalently,

$$s_i \geq F^{-1}(1 - z_i).$$

Note that this condition depends on the state only through z_i. Now let $\kappa(z)$ denote the expected value of $s_i - \mu$ when buyer i accepts the seller's offer which is accepted with probability z:

$$\kappa(z) = E\big[\tilde{s}_i - \mu \mid \tilde{s}_i \geq F^{-1}(1-z)\big].$$

Likewise, let $\lambda(z)$ denote the expected value of $s_i - \mu$ when buyer i rejects such an offer:

$$\lambda(z) = E\big[\tilde{s}_i - \mu \mid \tilde{s}_i < F^{-1}(1-z)\big].$$

Both $\kappa(z)$ and $\lambda(z)$ are independent of the state or the identity of the buyer who has made the decision, and by definition,

$$z\kappa(z) + (1-z)\lambda(z) = 0, \tag{A.16}$$

and $\kappa(z) \geq 0 \geq \lambda(z)$. If we denote by α the current state, then the next state is $\alpha + \kappa(z)$ when an offer that would be accepted with probability z is actually accepted, and $\alpha + \lambda(z)$ when it is rejected. Hence, the state goes up whenever an offer is accepted and goes down whenever it is rejected. Furthermore, since the initial state is zero and $\lim_{z\to 1} \kappa(z) = \bar{s} - \mu$ and $\lim_{z\to 0} \lambda(z) = \underline{s} - \mu$, if we let $C_t = [t(\underline{s} - \mu), t(\bar{s} - \mu)]$ for any $t = 0, 1, \ldots, I-1$, then the state at the completion of trades with the first t buyers must belong to C_t no matter what price offers those t buyers were given.

For any subset $J \subset I$ with $J \neq \emptyset$ and any permutation $\rho = (\rho_1, \ldots, \rho_J)$ over J, suppose that the seller has traded with the buyers in $I \setminus J$ and will now trade with buyers $\rho_1, \ldots, \rho_J$ in this order. Let J also denote the number of buyers in set J. The initial state (the state prior to the transaction with the first buyer ρ_1) is denoted α_0. We allow α_0 to be any point in C_{I-J} to reflect the outcome of the preceding transactions with $I - J$ buyers. Let $x_t(\alpha)$ denote the price offered to buyer ρ_t when the state is α, and $z_t(\alpha)$ be the probability that this offer is accepted.[23] Given the one-to-one correspondence between x and z in (A.15), we think that the seller controls $z_1, \ldots, z_J$ instead of $x_1, \ldots, x_J$. The *sequential pricing problem* (J, ρ, α_0) is a dynamic programming problem in which the seller, who trades with buyers $\rho_1, \ldots, \rho_J \in J$ in this order, maximizes revenue by controlling $z_1, \ldots, z_J$ given the the state variables $\alpha_0, \ldots, \alpha_{J-1}$.

[23]That is, x_t equals x_{ρ_t} in the previous notation.

Let $\Pi^*(J, \rho, \alpha_0)$ denote the maximized revenue in the sequential pricing problem (J, ρ, α_0). This can be obtained using backward induction as follows: Let $\pi_J(z_J, \alpha_{J-1})$ denote the expected revenue from the last buyer ρ_J in state α_{J-1} when the seller makes an offer that is accepted with probability z_J. In view of (A.15), it can be written as

$$\pi_J(z_J, \alpha_{J-1}) = x_J z_J = g(z_J) + c_{\rho_J} z_J \alpha_{J-1} + c_0 z_J,$$

where $g : [0,1] \to \boldsymbol{R}$ is defined by $g(z) = z\, F^{-1}(1-z)$. Let $\pi_J^*(\alpha_{J-1})$ denote the maximized value of $\pi_J(z_J, \alpha_{J-1})$:

$$\pi_J^*(\alpha_{J-1}) = \max_{z_J \in [0,1]} \pi_J(z_J, \alpha_{J-1}).$$

For $t = 1, \ldots, J-1$, the seller's expected revenue over periods $t, \ldots, J$ is recursively defined by

$$\pi_t(z_t, \alpha_{t-1}) = g(z_t) + c_{\rho_t} z_t \alpha_{t-1} + c_0 z_t + f_{t+1}(z_t, \alpha_{t-1}),$$

where

$$f_{t+1}(z_t, \alpha_{t-1}) = z_t\, \pi_{t+1}^*\Big(\alpha_{t-1} + \kappa(z_t)\Big) + (1 - z_t)\, \pi_{t+1}^*\Big(\alpha_{t-1} + \lambda(z_t)\Big)$$

is the expected revenue over periods $t+1, \ldots, J$ when he chooses z_t in period t, and then follows the optimal course of action in subsequent periods. The optimized value of $\pi_t(z_t, \alpha_{t-1})$ is denoted by

$$\pi_t^*(\alpha_{t-1}) = \max_{z_t \in [0,1]} \pi_t(z_t, \alpha_{t-1}).$$

We then have $\Pi^*(J, \rho, \alpha_0) = \pi_1^*(\alpha_0)$.

Step 2: Relationship between the Optimal Ordering and the Sequential Pricing Problem

Given $J \subset I$ and $i \in J$, denote by ρ^i the permutation over J obtained by putting i at the top and the remaining elements of J in the increasing order: $\rho_1^i = i$, $\rho_2^i = \min J \setminus \{i\}, \ldots, \rho_J^i = \max J \setminus \{i\}$.

Lemma A.2. *Suppose that $c_1 < \cdots < c_I$ and that for every subset $J \subset I$ and initial state $\alpha_0 \in C_{I-J}$,*

$$\Pi^*(J, \rho^{\min J}, \alpha_0) > \max_{\substack{i \in J \\ i \neq \min J}} \Pi^*(J, \rho^i, \alpha_0). \tag{A.17}$$

Then among all the sales schemes, the seller's expected revenue is maximized by the sequential scheme with a fixed order $(1, \dots, I)$.

With four buyers $I = 4$, for example, the above lemma can be illustrated as follows: Suppose first that for every pair of buyers and every initial state $\alpha_0 \in C_{I-2} = C_2$, we have $(1,2) \succ (2,1)$, $(1,3) \succ (3,1)$, $\dots$, $(3,4) \succ (4,3)$, where $\succ$ represents the ordering by the seller's revenue generated by the optimal contingent prices along the corresponding sequence of buyers. These comparisons determine the optimal ordering of the last two buyers depending on their identities. Suppose next that for any combination of three buyers and any initial state $\alpha_0 \in C_{I-3} = C_1$, we have

$$\begin{aligned} &(1,2,3) \succ (2,1,3),(3,1,2), \quad (1,2,4) \succ (2,1,4),(4,1,2), \\ &(1,3,4) \succ (3,1,4),(4,1,3), \quad (2,3,4) \succ (3,2,4),(4,2,3). \end{aligned}$$

The first two steps above determine the ordering of the last three buyers in the optimal scheme. Finally, suppose for $\alpha_0 \in C_0 = \{0\}$,

$$(1,2,3,4) \succ (2,1,3,4),\ (3,1,2,4),\ (4,1,2,3), \tag{A.18}$$

Then we can conclude that the optimal sequential scheme has a fixed order $(1,2,3,4)$.

Proof of Lemma A.2 Let $\sigma \in \Sigma$ be an optimal scheme. Take any history $h \in H_{I-2}$ at which two buyers i_1 and i_2 $(i_1 < i_2)$ remain. Since σ has a fixed order after h as noted earlier, the optimal pricing function associated with σ solves the sequential pricing problem $q = (J, \rho, \alpha_0)$, where $J = \{i_1, i_2\}$, $\rho = (i_1, i_2)$ or (i_2, i_1) and $\alpha_0 = \alpha^\sigma(h) \in C_{I-2}$. By assumption, then, trading with i_1 and i_2 in this order is better than the other way around. This implies that σ trades with the buyer with the smaller weight first whenever there remain two buyers. As an induction hypothesis, given any t $(2 \le t \le I-1)$, suppose that σ has a fixed, increasing order of dependence weights whenever there remain t buyers. Take any history h at which $t+1$ buyers $i_1, \dots, i_{t+1}$ $(i_1 < \cdots < i_{t+1})$ remain. σ then must solve the sequential pricing problem $q = (J, \rho, \alpha_0)$ with $J = \{i_1, \dots, i_{t+1}\}$ and $\alpha_0 = \alpha^\sigma(h) \in C_{I-t-1}$. Moreover, we have $\rho = \rho^i$ for some $i \in J$ since by the induction hypothesis, the t buyers after the first one are ordered monotonically in terms of their weights. By assumption, then, trading with $i_1, \dots, i_{t+1}$ in this order (*i.e.,* along $\rho^{\min J} = \rho^{i_1}$) is better than any other order ρ^i $(i \ne i_1)$. We can hence conclude that σ has a fixed, increasing order of weights after every history at which $t+1$ buyers remain. Hence, we have advanced the induction step and established that there exists an optimal non contingent scheme that has a fixed order $(1, \ldots, I)$.

Step 3: A sufficient condition for (A.17) in Lemma A.2.

Let a sequential pricing problem $q = (J, \rho, \alpha_0)$ and its solution $(z_1, \ldots, z_J)$ be given. Let $\alpha_t^+ = \alpha_{t-1} + \kappa(z_t(\alpha_{t-1}))$ be the state resulting when the offer $z_t(\alpha_{t-1})$ is accepted in state α_{t-1}, and $\alpha_t^- = \alpha_{t-1} + \lambda(z_t(\alpha_{t-1}))$ be the state resulting when it is rejected. Suppose that

$$z_t(\alpha_{t-1})\alpha_{t-1} < z_t(\alpha_{t-1})\, z_{t+1}(\alpha_t^+)\, \alpha_t^+ + (1 - z_t(\alpha_{t-1}))\, z_{t+1}(\alpha_t^-)\, \alpha_t^- \tag{A.19}$$

for every $\alpha_{t-1} \in C_{I-J+t-1}$ and $t = 1, \ldots, J$. Note that (A.19) captures the movement of the product $z_t\alpha_{t-1}$: The left-hand side gives the value of this product in period t, while the right-hand side is the period t expected value of the product in the following period. (A.19) requires that this value be increasing.

Lemma A.3. *Suppose that $c_1 < \cdots < c_I$. Take any subset $J \subset I$, permutation ρ over J such that $\rho_t > \rho_{t+1}$ for some $t = 1, \ldots, J-1$, and $\alpha_0 \in C_{I-J}$. Let ρ' be the alternative permutation over J that reverses the order of ρ_t and ρ_{t+1} but is otherwise the same: $\rho'_t = \rho_{t+1}$, $\rho'_{t+1} = \rho_t$ and $\rho'_n = \rho_n$ for $n \neq t,\ t+1$. If the solution z to the sequential pricing problem $q = (J, \rho, \alpha_0)$ satisfies (A.19), then the solution to the alternative sequential pricing problem $q' = (J, \rho', \alpha_0)$ yields a higher revenue: $\Pi^*(J, \rho', \alpha_0) > \Pi^*(J, \rho, \alpha_0)$.*

In other words, whenever a buyer with a smaller dependence weight comes immediately after a buyer with a larger weight, the revenue can be increased by reversing their order provided that the solution to the original problem satisfies (A.19). If (A.19) holds for the solution to every sequential pricing problem, hence, we can conclude that the conditions of A.2 hold. We will verify this in Step 5 below.

The intuition behind Lemma A.3 is as follows: As seen from (A.15), when the seller offers z_i to buyer i at state α, his revenue from buyer i can be written as:

$$x_i\, z_i = c_0 z_i + c_i z_i \alpha + z_i F^{-1}(1 - z_i). \tag{A.20}$$

It can be seen that the dependence weight c_i affects the stage revenue only through the product $z_i\alpha$ in (A.20). Therefore, when the product increases from one period to the next, the seller's revenue over the two periods is higher if the dependence weight in the second period is higher than that in the first. The formal proof consists of taking the pricing function for (J, ρ', α_0) under which the probability of acceptance by buyer $i = \rho'_t$ equals that by $j = \rho_t$ under z, and the probability of acceptance by $j = \rho'_{t+1}$ equals that by $i = \rho_{t+1}$ under z, and then showing that the increase in expected revenue equals the right-hand side minus the left-hand side of (A.19).

Proof of Lemma A.3 Let x be the pricing function associated with the solution z to the original sequential pricing problem (J, ρ, α_0). Given the alternative sequential pricing problem (J, ρ', α_0), let x' be the pricing function such that $x'_n(\alpha_{n-1}) = x_n(\alpha_{n-1})$ for any $\alpha_{n-1} \in C_{I-J+n-1}$ and $n \neq t$, $t+1$,

$$x'_t(\alpha_{t-1}) = x_t(\alpha_{t-1}) - (c_j - c_i)\, \alpha_{t-1} \quad \text{for any } \alpha_{t-1} \in C_{I-J+t-1},$$

and

$$x'_{t+1}(\alpha_t) = x_t(\alpha_t) + (c_j - c_i)\, \alpha_t \quad \text{for any } \alpha_t \in C_{I-J+t}.$$

It can then be verified that for any $\alpha_{t-1} \in C_{I-J+t-1}$,

$$\begin{aligned} z'_t(\alpha_{t-1}) &= P\Big(V_i^*(\tilde{s}_i \mid \alpha_{t-1}) \geq x'_i(\alpha_{t-1})\Big) \\ &= P\Big(c_0 + \tilde{s}_i + c_i\alpha_{t-1} \geq x_t(\alpha_{t-1}) - (c_j - c_i)\alpha_{t-1}\Big) \\ &= P\Big(c_0 + \tilde{s}_i + c_j\alpha_{t-1} \geq x_t(\alpha_{t-1})\Big) \\ &= P\Big(V_j(\tilde{s}_j \mid \alpha_{t-1}) \geq x_t(\alpha_{t-1})\Big) \\ &= z_t(\alpha_{t-1}). \end{aligned}$$

That is, the price offered to the tth buyer $\rho'_t = i$ under the alternative scheme is accepted with the same probability as that offered to the tth buyer $\rho_t = j$ under the original scheme. It can also be verified likewise that the price offered to the $t+1$th buyer j under the alternative scheme is accepted with the same probability as that offered to the tth buyer i under the original scheme: $z'_{t+1}(\alpha_t) = z_{t+1}(\alpha_t)$ for any $\alpha_t \in C_{I-J+t}$. Lemma A.1 then implies that for any $\alpha_{n-1} \in C_{I-J+n-1}$ and $n \neq t$, $t+1$, $z'_n(\alpha_{n-1}) = z_n(\alpha_{n-1})$.

Given any state $\alpha \in C_{I-J+t-1}$ at the beginning of period $t-1$, the seller's expected revenue from the alternative scheme minus that from the original scheme over the two periods t and $t+1$ is given by

$$\begin{aligned}
& z'_t(\alpha)\, x'_t(\alpha) + z'_t(\alpha)\, z'_{t+1}(\alpha + \kappa(z'_t(\alpha)))\, x'_{t+1}(\alpha + \kappa(z'_t(\alpha))) \\
& + (1 - z'_t(\alpha))\, z'_{t+1}(\alpha + \lambda(z'_t(\alpha)))\, x'_{t+1}(\alpha + \lambda(z'_t(\alpha))) \\
& - z_t(\alpha)\, x_t(\alpha) + z_t(\alpha)\, z_{t+1}(\alpha + \kappa(z_t(\alpha)))\, x_{t+1}(\alpha + \kappa(z_t(\alpha))) \\
& + (1 - z_t(\alpha))\, z_{t+1}(\alpha + \lambda(z_t(\alpha)))\, x_{t+1}(\alpha + \lambda(z_t(\alpha))) \\
& = (c_j - c_i) \Big[-z_t(\alpha)\, \alpha + z_t(\alpha)\, z_{t+1}(\alpha + \kappa(z_t(\alpha)))\, \alpha + \kappa(z_t(\alpha)) \\
& + (1 - z_t(\alpha))\, z_{t+1}(\alpha + \lambda(z_t(\alpha)))\, (\alpha + \lambda(z_t(\alpha))) \Big]
\end{aligned} \tag{A.21}$$

By (A.19), the quantity in the square brackets on the far right-hand side of (A.21) is > 0. Since the seller's revenue from any other period is the same under the two schemes, we obtain the desired conclusion.

Step 4: Solution to the Sequential Pricing Problem under the Uniform Distribution

We now derive an analytical solution to the sequential pricing problem when the buyers' private signals have a uniform distribution. Formally, suppose that for every $i \in I$, buyer i's private signal s_i has the uniform distribution over the interval $[\underline{s}, \bar{s}]$, *i.e.*,

$$F(s_i) = \frac{s_i - \underline{s}}{\Delta},$$

where $\Delta = \bar{s} - \underline{s}$. In this case, we have $\mu = (\underline{s} + \bar{s})/2$,

$$\kappa(z) = \frac{\Delta}{2}\,(1 - z) \quad \text{and} \quad \lambda(z) = -\frac{\Delta}{2}\, z.$$

Let $J \subset I$ be a subset of buyers and denote its cardinality also by J. Assume that $J = 2, \dots, I$, and take any ordering $\rho = (\rho_1, \dots, \rho_J)$ of them. Let $a_J = \Delta + \underline{s} + c_0$, $b_J = c_{\rho_J}$, and

$$a_t = \Delta + \frac{\underline{s} + c_0}{1 + \frac{1}{16} \sum_{k=t+1}^{J} b_k c_{\rho_k}}, \quad \text{and} \quad b_t = \frac{c_{\rho_t}}{1 + \frac{1}{16} \sum_{k=t+1}^{J} b_k c_{\rho_k}}$$

for $t = 1, \dots, J-1$. The following theorem describes the solution to the sequential pricing problem when it has an interior solution for every $t = 1, \dots, J$. Condition (A.22) guarantees that for any t, the optimal probability is an interior solution: $z_t(\alpha) \in (0, 1)$ for any $\alpha \in C_{I-J+t-1}$. Since $b_t \leq c_{\rho_t}$, this condition holds when $\underline{s} + c_0 < \Delta$ and the weights c_i are not so large. For example, when $\underline{s} + c_0 = 0$, (A.22) holds if $(I-1)\max_{i \in I} c_i < 2$.

Theorem A.4. *Suppose that each s_i has the uniform distribution over $[\underline{s}, \bar{s}]$. Take any subset J of buyers, any ordering $\rho = (\rho_1, \ldots, \rho_J)$ of them, and any initial state $\alpha_0 \in C_{I-J}$. If*

$$b_t < \frac{2}{\Delta(I - J + t - 1)} \min\{\Delta,\, 2\Delta - a_t\} \quad \text{for every } t = 1, \ldots, J, \tag{A.22}$$

the solution to the sequential pricing problem $q = (J, \rho, \alpha_0)$ is given by

$$z_t(\alpha) = \frac{1}{2\Delta}(a_t + b_t \alpha) \tag{A.23}$$

for any $\alpha \in C_{I-J+t-1}$ and $t = 1, \ldots, J$.

Proof of Theorem A.4 Suppose for simplicity that $J = I$ and that $\rho_t = t$ for every $t \in I$. Note that $g(z) = z\{\underline{s} + \Delta(1 - z)\}$ for the given uniform distribution F. Since

$$\frac{\partial \pi_I}{\partial z_I}(z_I, \alpha_{I-1}) = \bar{s} - 2\Delta z_I + c_I \alpha_{I-1} + c_0$$

is decreasing in z_I, the first-order condition yields the optimal solution $z_I(\alpha_{I-1}) = \frac{1}{2\Delta}(\bar{s} + c_0 + c_I \alpha_{I-1})$. The envelope theorem also implies that

$$\frac{\partial \pi_I^*}{\partial \alpha_{I-1}}(\alpha_{I-1}) = c_I z_I(\alpha_{I-1}).$$

As an induction hypothesis, suppose now that (A.23) holds for $i+1, \ldots, I$ ($i \leq I-1$) and that

$$\frac{\partial \pi_{i+1}^*}{\partial \alpha_i}(\alpha_i) = \sum_{j=i+1}^{I} c_j z_j(\alpha_i).$$

The expected revenue function for periods $i, \ldots, I$ can be written as

$$\pi_i(z_i, \alpha_{i-1}) = g(z_i) + z_i c_i \alpha_{i-1} + c_0 z_i + f_{i+1}(z_i, \alpha_{i-1}),$$

where

$$f_{i+1}(z_i, \alpha_{i-1}) = z_i\, \pi_{i+1}^*\Big(\alpha_{i-1} + \kappa(z_i)\Big) + (1 - z_i)\, \pi_{i+1}^*\Big(\alpha_{i-1} + \lambda(z_i)\Big).$$

It follows that

$$
\begin{aligned}
\frac{\partial \pi_i}{\partial z_i}(z_i, \alpha_{i-1}) &= \bar{s} - 2\Delta z_i + c_0 + c_i \alpha_{i-1} \\
&\quad + \sum_{j=i+1}^{I} c_j \int_{\alpha_{i-1}+\lambda(z_i)}^{\alpha_{i-1}+\kappa(z_i)} z_j(\alpha_i)\, d\alpha_i \\
&\quad - \frac{\Delta}{2} \sum_{j=i+1}^{I} c_j \Big\{ z_i z_j(\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) z_j(\alpha_{i-1} + \lambda(z_i)) \Big\} \\
&= \bar{s} - 2\Delta z_i + c_0 + c_i \alpha_{i-1} \\
&\quad + \sum_{j=i+1}^{I} c_j \int_{\alpha_{i-1}+\lambda(z_i)}^{\alpha_{i-1}+\kappa(z_i)} z_j(\alpha_i)\, d\alpha_i - \frac{\Delta}{2} \sum_{j=i+1}^{I} c_j z_j(\alpha_{i-1}) \\
&\quad + \sum_{j=i+1}^{I} c_j \left\{ \int_{\alpha_{i-1}+\lambda(z_i)}^{\alpha_{i-1}+\kappa(z_i)} z_j(\alpha_i)\, d\alpha_i - \frac{\Delta}{2} z_j(\alpha_{i-1}) \right\} \\
&= \bar{s} - 2\Delta z_i + c_0 + c_i \alpha_{i-1} + \sum_{j=i+1}^{I} \frac{\Delta}{16} b_j c_j (1 - 2 z_i),
\end{aligned}
\tag{A.24}
$$

where the second equality follows since z_j $(j = i+1, \dots, I)$ is by the induction hypothesis an affine function and since $z_i \kappa(z_i) + (1 - z_i)\lambda(z_i) = 0$:

$$
\begin{aligned}
& z_i z_j(\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) z_j(\alpha_{i-1} + \lambda(z_i)) \\
&= z_j\Big(z_i(\alpha_{i-1} + \kappa(z_i)) + (1 - z_i)(\alpha_{i-1} + \lambda(z_i)) \Big) \\
&= z_j(\alpha_{i-1}).
\end{aligned}
\tag{A.25}
$$

Since $\dfrac{\partial \pi_i}{\partial z_i}$ is decreasing in z_i, the first-order condition yields the optimal solution

$$
z_i(\alpha_{i-1}) = \frac{1}{2\Delta} \, \frac{\bar{s} + c_0 + c_i \alpha_{i-1} + \sum_{j=i+1}^{I} \Delta b_j c_j / 16}{1 + \sum_{j=i+1}^{I} b_j c_j / 16} = \frac{1}{2\Delta} \, (a_i + b_i \alpha_{i-1}).
$$

Furthermore, using (A.25) again, we see that

$$\begin{aligned}\frac{\partial f_{i+1}}{\partial \alpha_{i-1}}(z_i,\alpha_{i-1}) &= z_i \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1}+\kappa(z_i)) + (1-z_i)\frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1}+\lambda(z_i))\\ &= \sum_{j=i+1}^{I} c_j \Big\{ z_i z_j(\alpha_{i-1}+\kappa(z_i)) + (1-z_i) z_j(\alpha_{i-1}+\lambda(z_i)) \Big\}\\ &= \sum_{j=i+1}^{I} c_j z_j \Big(\alpha_{i-1} + z_i\kappa(z_i) + (1-z_i)\lambda(z_i) \Big)\\ &= \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1}).\end{aligned} \tag{A.26}$$

Hence, the envelope theorem implies that

$$\frac{\partial \pi^*_i}{\partial \alpha_{i-1}}(\alpha_{i-1}) = c_i z_i(\alpha_{i-1}) + \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1}) = \sum_{j=i}^{I} c_j z_j(\alpha_{i-1}).$$

This advances the induction step and completes the proof.

Step 5: Conditions of Theorem 2 imply (A.19) and (A.22).

Write $q_t = \Big(1 + \frac{1}{16}\sum_{k=t+1}^{I} b_k c_{\rho_k}\Big)^{-1} > 1$. (A.22) holds if

$$b_t < \frac{2}{I-1} \min\Big\{1,\, 1 - \frac{\underline{s}+c_0}{\Delta}\, q_t\Big\}.$$

Since $b_t < \max_{i\in I} c_i$, and $\max_{i\in I} c_i < \frac{2}{I-1} - \beta$ by (4), this inequality holds if δ is such that $\delta < \frac{I-1}{2}\beta$.

We next show that (A.19) holds for a sufficiently small δ. Note first that $\max_i c_i^2 < \frac{4}{(I-1)^2} - \beta^2$ by the first condition in (4) and the choice of β. It follows from this and the second condition in (4) that

$$\frac{c_i}{c_j} \le 1 - \frac{c_i^2}{16} + \frac{1}{4(I-1)^2} \quad \text{for any } i \ne j. \tag{A.27}$$

Since $\frac{b_t}{b_{t+1}} < \frac{c_{\rho_t}}{c_{\rho_{t+1}}}$ and $b_t \le c_{\rho_t}$, (A.27) implies that

$$\frac{b_t}{b_{t+1}} < 1 - \frac{b_t^2}{16} + \frac{1}{4(I-1)^2} \quad \text{for } t = 1, \dots, I-1. \tag{A.28}$$

Now, since z_t is an affine function by Theorem A.4, we have for any α,

$$\begin{aligned}
&z_t(\alpha)\, z_{t+1}\Big(\alpha + \kappa(z_t(\alpha))\Big) + (1 - z_t(\alpha))\, z_{t+1}\Big(\alpha + \lambda(z_t(\alpha))\Big) \\
&= z_{t+1}\Big(\alpha + z_t(\alpha)\, \kappa(z_t(\alpha)) + (1 - z_t(\alpha))\, \lambda(z_t(\alpha))\Big) \\
&= z_{t+1}(\alpha),
\end{aligned} \tag{A.29}$$

where the second equality follows from the definitions of κ and λ. Using (A.23) and (A.29), we see that (A.19) holds at any state in any period if

$$\begin{aligned}
\{16(b_{t+1} - b_t) - b_t^2 b_{t+1}\}\, \alpha^2 + 2\{8(a_{t+1} - a_t) + b_t b_{t+1}(\Delta - a_t)\}\, \alpha \\
+ a_t b_{t+1}(2\Delta - a_t) > 0 \\
\text{for any } \alpha \in C_{I-J+t-1} \text{ and } t = 1, \dots, J.
\end{aligned} \tag{A.30}$$

Since the left-hand side of (A.30) is quadratic in α, we have the following two cases to consider to determine if (A.30) holds for any $\alpha \in C_{I-J+t-1}$.

Case 1. $16(b_{t+1} - b_t) - b_t^2 b_{t+1} > 0$.

In this case, (A.30) holds if the determinant is strictly negative:

$$\{8(a_{t+1} - a_t) + b_t b_{t+1}(\Delta - a_t)\}^2 < a_t b_{t+1}(2\Delta - a_t)\{16(b_{t+1} - b_t) - b_t^2 b_{t+1}\}. \tag{A.31}$$

Using $a_t = \Delta + q_t(\underline{s} + c_0)$, we can rewrite (A.31) as

$$\begin{aligned}
&\{8(q_{t+1} - q_t) - b_t b_{t+1} q_t\}^2 \Big(\frac{\underline{s} + c_0}{\Delta}\Big)^2 \\
&< \Big\{1 - \Big(\frac{\underline{s} + c_0}{\Delta}\Big)^2\Big\} b_{t+1}\{16(b_{t+1} - b_t) - b_t^2 b_{t+1}\}.
\end{aligned}$$

Since $16(b_{t+1} - b_t) - b_t^2 b_{t+1} > 0$ by assumption, this holds if $|\frac{\underline{s}+c_0}{\Delta}| < \delta$ for a sufficiently small δ.

Case 2. $16(b_{t+1} - b_t) - b_t^2 b_{t+1} \leq 0$.

In this case, (A.30) holds for any $\alpha \in C_{I-J+t-1}$ if it holds at $\alpha = -\frac{\Delta}{2}\,(I - 1)$ and $\alpha = \frac{\Delta}{2}\,(I - 1)$ since $C_{I-J+t-1} \subset C_{I-1} = [-\frac{\Delta}{2}\,(I - 1),\, \frac{\Delta}{2}\,(I - 1)]$. These conditions can be summarized as:

$$\begin{aligned}
&\{16(b_{t+1} - b_t) - b_t^2 b_{t+1}\}\, \frac{1}{4}\,(I - 1)^2 + \frac{1}{\Delta^2}\, a_t(2\Delta - a_t) b_{t+1} \\
&> \frac{1}{\Delta}\,(I - 1)\, \Big|8(a_{t+1} - a_t) + b_t b_{t+1}(\Delta - a_t)\Big|.
\end{aligned} \tag{A.32}$$

Using $a_t = \Delta + q_t(\underline{s} + c_0)$ again, we can rewrite (A.32) as

$$
\begin{aligned}
&1 - \frac{b_t}{b_{t+1}} - \frac{b_t^2}{16} + \frac{1}{4(I-1)^2} \\
&> \frac{\underline{s}+c_0}{\Delta} \frac{1}{4(I-1)} \left\{ \frac{1}{I-1} \frac{\underline{s}+c_0}{\Delta} + \frac{1}{b_{t+1}} \Big| 8(q_{t+1} - q_t) - b_t b_{t+1} q_t \Big| \right\}.
\end{aligned}
$$

By (A.28), this holds if $|\frac{\underline{s}+c_0}{\Delta}| < \delta$ for a sufficiently small δ.

Proof of Proposition 1 By Lemma A.5 below, the conclusion follows if it is shown that

$$\sum_{t=1}^{I} (t-1)\, c_t > \bar{c} \sum_{t=1}^{I} (t-1).$$

This holds since

$$
\begin{aligned}
\frac{\sum_{t=1}^{I} (t-1)\, c_t}{\sum_{t=1}^{I} (t-1)} &= \frac{\sum_{t=2}^{I} c_t + \sum_{t=3}^{I} c_t + \cdots + (c_{I-1} + c_I) + c_I}{(I-1) + (I-2) + \cdots + 2 + 1} \\
&> \frac{\sum_{t=2}^{I} c_t}{I-1} > \frac{\sum_{t=1}^{I} c_t}{I} = \bar{c}.
\end{aligned}
$$

Lemma A.5. *Let $\bar{\alpha}_0 = \underline{\alpha}_0 = 0$, and for $t = 2, \ldots, I$, let $\bar{\alpha}_{t-1}$ and $\underline{\alpha}_{t-1}$ be the states at the beginning of period t when the buyers $1, \ldots, t-1$ have all accepted, and when they have all rejected, respectively. Then $\bar{\alpha}_t$ and $\underline{\alpha}_t$ are given by*

$$\bar{\alpha}_t = \frac{t}{4}\Delta + o(1) \quad \textit{and} \quad \underline{\alpha}_t = -\frac{t}{4}\Delta + o(1),$$

where for any $m = 0, 1, \ldots$, $o(\varepsilon^m)$ is any term such that $\lim_{\varepsilon \to 0} |o(\varepsilon^m)|/\varepsilon^m = 0$. Furthermore, the probability that all buyers accept equals

$$\prod_{t=1}^{I} z_t(\bar{\alpha}_{t-1}) = \left(\frac{1}{2}\right)^I \left\{ 1 + \frac{1}{4} \sum_{t=1}^{I} (t-1)\, c_t \right\} + o(\varepsilon),$$

and the probability that all buyers reject equals

$$\prod_{t=1}^{I} (1 - z_t(\underline{\alpha}_{t-1})) = \left(\frac{1}{2}\right)^I \left\{ 1 + \frac{1}{4} \sum_{t=1}^{I} (t-1)\, c_t \right\} + o(\varepsilon)$$

Proof. Note first that $\bar{\alpha}_t$ and $\underline{\alpha}_t$ $(t = 1, \ldots, I-1)$ are recursively defined by

$$\bar{\alpha}_t = \bar{\alpha}_{t-1} + \kappa(z_t(\bar{\alpha}_{t-1})), \quad \text{and} \quad \underline{\alpha}_t = \underline{\alpha}_{t-1} + \lambda(z_t(\underline{\alpha}_{t-1})).$$

The probabilities that all buyers accept and that they all reject can then be expressed as

$$\prod_{t=1}^{I} z_t(\bar{\alpha}_{t-1}) \quad \text{and} \quad \prod_{t=1}^{I} (1 - z_t(\underline{\alpha}_{t-1})),$$

respectively. Note next that $a_t = \Delta$ $(t = 1, \ldots, I)$ when $\underline{s} + c_0 = 0$. We also have $b_I = c_I$ by definition, and can also show by induction that

$$b_t = c_t + o(\varepsilon^2). \tag{A.33}$$

for $t = 1, \ldots, I-1$.[24] For $t = 1$, we have

$$\bar{\alpha}_1 = \kappa(z_1(\bar{\alpha}_0)) = \frac{\Delta}{2}\Big(1 - \frac{a_1}{2\Delta}\Big) = \frac{\Delta}{4} + o(1).$$

As an induction hypothesis, suppose that for $t = 2, \ldots, I-1$,

$$\bar{\alpha}_{t-1} = \frac{t-1}{4}\,\Delta + o(1).$$

Then we can use (A.33) to conclude that

$$\begin{aligned}
\bar{\alpha}_t &= \bar{\alpha}_{t-1} + \kappa(z_t(\bar{\alpha}_{t-1})) \\
&= \bar{\alpha}_{t-1} + \frac{\Delta}{2}\Big\{1 - \frac{1}{2\Delta}\,(a_t + b_t\bar{\alpha}_{t-1})\Big\} \\
&= \bar{\alpha}_{t-1} + \frac{1}{4}\Big\{\Delta - \bar{\alpha}_{t-1}c_t + o(\varepsilon)\Big\} \\
&= \frac{t}{4}\,\Delta + o(1),
\end{aligned}$$

and hence the induction step is advanced. The proof for $\underline{\alpha}_t$ is similar. It follows that when the buyers $\rho_1, \ldots, \rho_{t-1}$ have all accepted, the probability that buyer ρ_t

[24]We have $b_I = c_I$, and $b_{I-1} = c_{I-1} - c_{I-1}\frac{c_I^2}{1+c_I^2} = c_{I-1} + o(\varepsilon^2)$. For $t = 1, \ldots, I-2$, if $b_k = c_{\rho_k} + o(\varepsilon^2)$ for $k = t+1, \ldots, I-1$, then

$$b_t = c_t - c_t\,\frac{\sum_{k=t+1}^{I} b_k c_k}{1 + \sum_{k=t+1}^{I} b_k c_k} = c_t + o(\varepsilon^2).$$

We can also show that $a_t = \Delta + o(\epsilon)$ when $\underline{s} + c_0 \neq 0$.

also accepts is given by

$$
\begin{aligned}
z_t(\bar{\alpha}_{t-1}) &= \frac{1}{2\Delta}\left(a_t + b_t\bar{\alpha}_{t-1}\right) \\
&= \frac{1}{2\Delta}\left[\Delta + o(\varepsilon) + \frac{c_t}{4}(t-1)\Delta + o(\varepsilon)\right] \\
&= \frac{1}{2}\left[1 + \frac{c_t}{4}(t-1)\right] + o(\varepsilon).
\end{aligned}
$$

Hence, the probability that every buyer accepts can be computed as

$$
\begin{aligned}
\prod_{t=1}^{I} z_t(\bar{\alpha}_{t-1}) &= \left(\frac{1}{2}\right)^I \prod_{t=1}^{I}\left\{1 + \frac{c_t}{4}\,(t-1)\right\} + o(\varepsilon) \\
&= \left(\frac{1}{2}\right)^I \left\{1 + \sum_{t=1}^{I} \frac{c_t}{4}\,(t-1)\right\} + o(\varepsilon).
\end{aligned}
$$

The probability that every buyer rejects can be computed in a similar manner.

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