Information Feedback in a Dynamic Tournament*

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Abstract

This paper studies the problem of information revelation in a multi-stage tournament where the agents’ effort in each stage gives rise to a stochastic performance signal privately observed by the principal. The principal controls the agents’ effort incentive through the use of a feedback policy, which transforms his private information into a public announcement. The optimal feedback policy is one that maximizes the agents’ expected effort. The paper identifies when the principal should use the no-feedback policy that reveals no information, or the full-feedback policy that reveals all his information.

Key words: tournament, mechanism, information revelation, Jensen’s inequality.

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1 Introduction

As a prominent form of relative performance evaluation, tournaments have attracted considerable attention in economic theory. The main focus of the theory is on the size and allocation of rewards that would maximize the performance of the competing agents, and on the comparison of the relative incentive schemes against more general forms of contracts. Beginning with the seminal work of Lazear and Rosen (1981), a partial list of the literature on this subject includes Green and Stokey (1983), Nalebuff and Stiglitz (1983), Glazer and Hassin (1988), Gradstein and Konrad (1999), Moldovanu and Sela (2001), and others. In most models, a tournament is described as a static mechanism in which the agents’ one-time effort decision determines their performance and hence the winner. In reality, however, many tournaments are more appropriately described as dynamic games: Agents make sequential effort decisions in multiple stages and the winner is determined by their overall performance. One important consideration when designing a tournament as a dynamic mechanism concerns the control of information during the course of play. In other words, the design of a dynamic tournament should include the strategic planning of what information to make available to the participants at what stage. The mode of such information revelation will have a significant impact on the participants’ effort incentive. This point is well exemplified by a tournament for job promotion within a firm: First, such a tournament is dynamic in nature and spans multiple stages. Second, workers’ performance is often measured by subjective criteria such as leadership, originality, ability to work in teams, etc. Such information is most appropriately described as private information of their superior or the firm’s personnel department, and the latter communicates this information back to the workers as a way of providing motivation. Research on performance management well recognizes that inducement of the work incentive requires careful designing of information feedback.\(^1\)

In this paper, we formulate a model of a dynamic tournament in which the principal receives private information about agents’ performance, and then reveals as a feedback some or all of his information to the agents. The analysis is dual to that in the standard contest literature in that we fix prizes and focus exclusively on the effects of information. While strategic transmission of private information is a much studied subject in economic theory, no general understanding exists about how the designer of a mechanism should incorporate his own private information

\(^1\)See, for example, Williams (1998).
into the mechanism. Existing theories provide varying intuitions as follows.

In auction theory, the so-called linkage principle by Milgrom and Roberts (1982) asserts that under the affiliated distribution of signals, the seller’s expected revenue is the highest when he is committed to revealing all of his private information to the bidders.\(^2\) In a related framework, Milgrom (1981) shows that the seller of a good maximizes his payoff by revealing all his private information to the buyer if it is affiliated with the quality of his good. In some other situations, however, it is shown that the intuition furnished by the linkage principle fails to hold: Kaplan and Zamir (2000) analyze the problem of an auctioneer privately informed about bidders’ valuations. In an independent private values framework, they find that the auctioneer is better off revealing the maximum of the valuations than fully revealing his information. In a model of twice-repeated common-value auctions with affiliated signals, de-Frutos and Rosenthal (1998) show that the auctioneer’s expected revenue (over two auctions) is lower when information about stage 1 bids is made public than when it is not.\(^3\)

The literature on dynamic models of a race also provides a closely related observation in the discussion of the closed- and open-loop formats.\(^4\) The open-loop format reveals no information to the players during a competition, whereas the closed-loop format reveals the competitors’ positions publicly and instantaneously. It is often argued that the players tend to slack off in the closed-loop format since, when one player has a lead over the others, the followers cannot catch up with the leader (in expected terms) by making the same level of effort as him. For example, Fudenberg et al. (1983) demonstrate the phenomenon of \(\epsilon\)-preemption, where players stop making effort as soon as one of them establishes a small lead over others.

It is important to understand that information feedback has two separate effects on the agents’ incentives. First, the revealed information influences the agents’ incentives by changing their beliefs. This is true irrespective of whether the principal’s private information is given exogenously as in the case of the linkage principle, or is generated endogenously by the agents’ own actions as in our model. We call this the \textit{ex post} effect of information feedback. On the other hand, when the private information is generated endogenously, each agent will choose their actions strategically

\(^2\) A probability distribution is \textit{affiliated} if the joint density function is log-supermodular.

\(^3\) Perry and Reny (1999) report the failure of the linkage principle in a multi-object auction based on an entirely different logic.

\(^4\) See, for example, Harris and Vickers (1985), and Fudenberg \textit{et al.} (1983). Radner (1985) also makes a related observation in the context of a repeated principal-agent game.
so as to influence the content of the revealed information. For example, agents may exert extra effort in early stages to take the leading position and discourage opponents. We call this the strategic effect of information feedback. One interpretation of the finding of de-Frutos and Rosenthal (1998) is that these two effects may offset each other.

In our model of a multi-stage tournament, agents’ performance in each stage is stochastically related to their effort in that stage. The principal privately observes their performance realization after each stage, and reveals some or all of his private information to the agents before the next stage. The principal’s feedback policy transforms the raw observation of the agents’ performance into a public announcement. In our terminology, the closed-loop and open-loop formats described above correspond to the full-feedback and no-feedback policies, respectively.\(^5\) The principal is free to choose any feedback policy and publicly announces its use before the tournament. For example, he may declare the use of a hybrid policy that reveals full information for some signal realizations but no information for others.\(^6\) We assume that the principal is committed to his feedback policy for any realization of the private signal. The optimal feedback policy is one that maximizes the principal’s payoff which is an increasing function of the agents’ expected efforts. As discussed below, we find that whether he should reveal more information or not depends critically on the functional form of the agents’ disutility of effort.

A more detailed description of our model is as follows: Two agents compete in a tournament over \(T\) stages. The agent with the higher performance at the end of stage \(T\) wins and is awarded a prize of a fixed value such as a promotion to a higher job rank. In every stage \(t\), each agent \(i\) chooses an effort level \(a_i^t\), which is observed by neither the principal nor his opponent. The agents’ cost function of effort is time-separable and can be expressed as the sum of stage-cost functions, which are assumed to be all strictly convex. The score in stage \(t\) is the difference between the performance levels of the two agents and equals the sum of the difference between their effort levels and a random noise term. The principal privately observes the score, and makes a public announcement about it at the end of stage \(t\). Conditional on the announcement, the agents update their inference about the score and decide on their effort levels in subsequent stages. We study how the choice of a feedback policy affects the agents’ effort levels in a pure perfect Bayesian equilibrium (PBE)

\(^5\)Alternatively, the no-feedback policy can be interpreted as the simultaneous implementation of multiple one-shot tournaments.

\(^6\)Under such a policy, of course, “no announcement” also has an informational content.
of this dynamic game.

The distinguishing features of the present model as compared with the models of auctions and dynamic races mentioned above are as follows. First, we explicitly analyze the strategic effect of information feedback. In other words, each agent in our model chooses his effort level fully taking into consideration its effect on the principal’s announcement and ultimately on future competition. Second, an agent’s effort is a continuous variable chosen from the set of real numbers. No matter how large the lead may be, hence, the follower can still make a very high effort and leapfrog his opponent. This possibility is often precluded in the models of a race. Third, agents are symmetric in ability and performance is a noisy outcome of effort. In other words, agents in the present model do not have any signaling motive in choosing their actions.

The paper presents sufficient conditions for the existence of a perfect Bayesian equilibrium (PBE) and derives effort levels on the equilibrium path. Specifically, we show that a pure PBE exists if the noise component of the performance score is sufficiently large. In particular, it is worth noting that the level of noise required can be taken independent of the feedback policy. We then proceed to characterize the optimal feedback policy. In short, revealing more information is better for the principal when the marginal cost of effort is concave, and the converse is true when the marginal cost is convex. More specifically, the following observations are made for the basic model with two stages ($T = 2$): When the stage 2 marginal cost function of effort is convex, the no-feedback policy is optimal in the class of feedback policies that admit a symmetric PBE. On the other hand, the full-feedback policy is optimal in the same class when the marginal cost function is concave. When the two agents’ efforts are sufficiently complementary to each other in the principal’s payoff function, the no-feedback and full-feedback policies are also optimal within the wider class of feedback policies that induce a possibly asymmetric PBE.

The intuition for the above conclusions for the symmetric PBE is as follows: As is standard, the agents’ effort choice in each stage balances its marginal disutility with the marginal increment in the probability of winning. It can be readily verified that the marginal increment in the probability of winning from stage 2 effort equals the conditional expectation of a function of the stage 1 score given the public announcement. Note in particular that the principal’s feedback policy determines the coarseness of the conditioning filtration. As for the stage 2 effort, its expected value equals the (unconditional) expectation of the above conditional expectation.
inverted by the stage 2 marginal cost function. When (the inverse of) the stage 2 marginal cost function is convex or concave, therefore, Jensen’s inequality yields an ordering over various feedback policies according to the expected effort they induce. As for the stage 1 effort, on the other hand, it can be verified that its marginal disutility in equilibrium equals the expected marginal disutility from stage 2 effort.\footnote{Although intuitive, this cannot be assumed \textit{a priori} because of the strategic effect of stage 1 effort mentioned above. For example, agents may choose to exert larger effort in stage 1 in order to preempt the leading position.} In other words, the marginal disutility of stage 1 effort is set equal to the unconditional expectation of the above function of the stage 1 score, which is independent of the filtration by the law of iterated expectation. Therefore, the stage 1 effort is constant under any feedback policy. Combination of these observations leads to the desired conclusion.

Given these observations, one interesting question concerns the optimal feedback policy when the marginal cost of effort is neither convex nor concave on the relevant domain. We attempt to answer this question by examining the marginal cost function having a single reflection point at which its curvature changes from convex to concave. Our candidate feedback policy reveals full information when the absolute value of the score is less than some threshold, and reveals nothing (other than the fact that the threshold has been exceeded) otherwise. We show that such a feedback policy outperforms the full-feedback policy. A similar argument proves that no-feedback policy is dominated by the feedback policy that only reveals whether or not the score has exceeded some threshold.

We next generalize our analysis to a $T$-stage tournament in which information feedback has many more dimensions than in the two-stage model. In particular, a feedback policy in a $T$-stage model is a contingent plan which determines not only the amount of information revelation but also its timing. For example, the principal may choose to reveal the stage 1 score before stage 3 if the stage 2 score is in some range, but withhold it until stage 4 otherwise. For a large noise level, we prove that a symmetric PBE in the $T$-stage model exists when a feedback policy is \textit{even} in the sense that it reveals no information on the identity of the leader. We then show that the qualitative conclusions of the two-stage model extend to the $T$-stage environment for this class of feedback policies. Specifically, the no-feedback policy is optimal among even feedback policies when the stage $t$ marginal cost function is convex for $t = 2, \ldots, T$. When the stage $t$ marginal cost function is concave,
on the other hand, the feedback policy that reveals the absolute value of the lead after every period is optimal in the class of even policies. Given that no even policy reveals more information than the latter feedback policy, this conclusion corresponds to that concerning the full-feedback policy in the two-stage model.

As seen from the above discussion, the optimal feedback policy often takes a simple but extreme form. What is important to note is the sensitivity of the optimal solution to the specification of the parameters of the model. For example, revealing no information is optimal in some cases, but is least desirable in others. Such sensitivity may in part explain the variations in the intuitions obtained from the existing models of information revelation as discussed above. On the other hand, it is also interesting to note that the mode of information feedback is irrelevant under the common assumption of quadratic cost functions. In this case, not only are the no-feedback and full-feedback policies optimal, but so is any feedback policy.

The literature on a dynamic principal-agent relationship makes some related observations on the agent’s effort incentive and information. In the analysis of a repeated principal-agent game with a public performance signal, Radner (1985) considers a review strategy for the principal that evaluates the agent’s performance at the end of each review phase that spans a large number of periods. He notes that inefficiency is inevitable as the agent relaxes near the end of the review phase if he realizes that his effort no longer influences the outcome of the review. In a two-stage principal-agent relationship with public performance information, Rogerson (1985) shows that the optimal stage 2 contract conditions on the stage 1 outcome. In his model, a condition involving the third derivative of the agent’s utility function determines whether the expected wage rises or falls over time.

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The paper is organized as follows: In the next section, we develop the basic framework for the two-stage model. Section 3 characterizes a PBE and provides sufficient conditions for its existence. In Section 4, the analysis of the optimal feedback policy is given. An extension to the general \( T \)-stage model is presented in Section 5. We conclude in Section 6.

2 Discrete Effort Levels

3 Model of a Tournament

Two risk neutral agents \( i = 1, 2 \) compete in two stages. In each stage, the agents’ effort gives rise to a stochastic “score,” which indicates their relative performance. At the end of stage 2, the principal aggregates the scores from both stages to determine the winner.\(^{10}\)

Formally, suppose that agent \( i \)'s effort \( a_i^t \) in stage \( t \) is chosen from the set \( R_+ \) of non-negative real numbers. The stage \( t \) score \( x_t \) is a random variable whose distribution depends on the effort levels \( a_1^t \) and \( a_2^t \) of both agents in stage \( t \). More specifically, we assume that \( x_t = a_1^t - a_2^t + \zeta_t \) for a real-valued random variable \( \zeta_t \). In other words, the score \( x_t \) represents agent 1's lead over agent 2, and is stochastically related to the difference between their effort levels. Let \( \phi_t \) be the density of \( \zeta_t \) over \( R \), and denote by \( \Phi_t \) the corresponding cumulative distribution. We assume that \( \phi_t \) is strictly positive and twice continuously differentiable, and symmetric around zero in the sense that \( \phi_t(x) = \phi_t(-x) \) for any \( x \in R \). We also assume that \( \zeta_1 \) and \( \zeta_2 \) are independent. Note that the density of \( x_t \) under the action profile \( a_t = (a_1^t, a_2^t) \) is given by

\[
\phi_t(x_t - a_1^t + a_2^t).
\]

The (aggregate) score \( x \) is the sum of scores in stages 1 and 2: \( x = x_1 + x_2 \). Agent 1 wins if \( x > 0 \), and agent 2 wins if \( x < 0 \). Each agent wins with equal probability in the (probability zero) event of a tie \( x = 0 \).

Each agent derives one unit of positive utility from winning the prize (e.g., promotion to a higher job rank), and incurs disutility from effort. The cost of effort in stage \( t \) is described by a twice differentiable cost function \( c_t : R_+ \rightarrow R_+ \). Accordingly, agent \( i \)'s overall utility equals \( 1 - \sum_{t=1}^{2} c_t(a_i^t) \) if he wins, and

\(^{10}\)While this is a special case of the more general model discussed in Section 5, the two-stage setting allows for a more concise presentation of the results under more permissive conditions.
\[- \sum_{t=1}^{2} c_t(a_i^t) \text{ otherwise.} \] The principal’s payoff, on the other hand, is a function of each agent’s effort levels in the two stages: \(V(a_1^1, a_2^1, a_2^2)\). The function \(V: \mathbb{R}_+^4 \rightarrow \mathbb{R}\) is assumed to be increasing (\(V(\hat{a}) \geq V(a)\) if \(\hat{a}_t^i \geq a_t^i\) for each \(t, i = 1, 2\)) and symmetric with respect to the agents (\(V(\hat{a}) = V(a)\) if \(\hat{a}_1^i = a_2^i\) and \(\hat{a}_2^i = a_1^i\) for \(t = 1, 2\)). Note that the principal’s payoff may in general contain more information than his private signal \(x\) about the agents’ efforts. In line with our assumption that the winner is determined based only on \(x\), we suppose that the principal observes his payoff only after the winner has been determined. Each agent’s effort \(a_i^t\) is his private information and observed by neither the principal nor the other agent. On the other hand, the principal privately observes the score \(x_t\) in each stage \(t\) and reveals either whole or part of his private information \(x_1\) after stage 1. Specifically, suppose that the principal makes a public announcement \(y\) about \(x_1\) at the end of stage 1. Formally, a feedback policy (or simply a policy) is a pair of the set of possible announcements \(Y\), and a measurable mapping \(f: \mathbb{R} \rightarrow Y\), which chooses the announcement \(y = f(x_1)\) as a function of the score \(x_1\). For simplicity, the reference to \(Y\) will be omitted and the mapping \(f\) alone will be called a feedback policy in what follows. It is understood that \(Y = \{f(x_1) : x_1 \in \mathbb{R}\}\) so that \(f\) is a surjection. The announcement \(y\) is credible in the sense that the principal publicly announces his feedback policy \(f\) before the tournament begins and uses it to generate \(y\) for any signal \(x_1\). The principal’s objective is to maximize his expected payoff by controlling \(f\). Although we will restrict our analysis to deterministic feedback policies, the paper’s conclusions hold even when we allow for stochastic feedback policies, which choose the announcement \(y\) as a function of \(x_1\) and some (exogenous) random variable.

Little restriction is placed on the nature of the public announcement \(y\). For example, each announcement \(y \in Y\) may simply contain the name of the leader, or it may be an interval in \(\mathbb{R}\) which indicates the range of \(x_1\).

As mentioned in the Introduction, some simple feedback policies will play an import role in our analysis. In particular, the no-feedback policy sends the same message regardless of \(x_1\), and the full-feedback policy reveals \(x_1\) completely. Between these two are numerous policies that reveal an intermediate amount of information. For example, we will later discuss the hybrid policies which reveal full information when the score is within some range \((-b, b)\) \((b > 0)\), but nothing otherwise: \(Y = \)}
\((-b, b) \cup \{N\}\), and
\[
  f(x_1) = \begin{cases} x_1 & \text{if } |x_1| < b, \\ N & \text{otherwise.} \end{cases}
\]

Of course, the agents hearing the announcement \(N\) know that \(|x_1| \geq b\). Given any announcement \(y \in Y\), let \(f^{-1}(y) = \{x_1 \in R : f(x_1) = y\}\) denote the inverse image of the (singleton) set \(\{y\}\) under \(f\). In what follows, we will restrict attention to feedback policies that satisfy the following regularity condition: A feedback policy \(f\) is regular if for any \(y \in Y\), \(f^{-1}(y) \subset R\) either has positive (Lebesgue) measure, or is countable.

The above hybrid policy, for example, is regular since \(f^{-1}(N) = (-\infty, -b) \cup (b, \infty)\) has positive measure and for \(x_1 \in (-b, b)\), \(f^{-1}(x_1) = \{x_1\}\) is countable.\(^{11}\)

Given any policy \(f\), agent \(i\)'s history \(h_i\) after stage 1 is the information available to agent \(i\) at the end of stage 1: \(h_i\) consists of his own effort choice \(a_i^1\), and the public announcement \(y\) by the principal. Agent \(i\)'s (pure) strategy \(\sigma_i\) is a pair \((\sigma_i^1, \sigma_i^2)\), where \(\sigma_i^1 \in R_+\) is the effort choice for stage 1, and \(\sigma_i^2 : R_+ \times Y \rightarrow R_+\) is a mapping that specifies the stage 2 effort after each possible history \(h_i^1\).

Given the strategy profile \(\sigma\), let \(\pi_i^2(a_i^2 \mid \sigma, h_i^1)\) denote agent \(i\)'s expected payoff in stage 2 (payoff from the possible prize minus the cost of stage 2 effort) when he chooses \(a_i^2\) in stage 2, his history in stage 1 is \(h_i^1\), and agent \(j\) plays according to the strategy \(\sigma_j\) in both stages. Likewise, let \(\pi_i^1(a_i^1 \mid \sigma)\) denote agent \(i\)'s overall expected payoff when he chooses \(a_i^1\) in stage 1 and plays according to \(\sigma_j\) in stage 2, and agent \(j\) plays according to \(\sigma_j\) in both stages. Given that the distribution \(\phi_1\) has full support, there is no ambiguity about the agents' belief. For this reason, we define an equilibrium simply in terms of a strategy profile. Specifically, a strategy profile \(\sigma = (\sigma^1, \sigma^2)\) is a (pure) perfect Bayesian equilibrium (PBE) if for \(i = 1, 2\),
\[
  \pi_i^1(\sigma_i^1 \mid \sigma) \geq \pi_i^1(a_i^1 \mid \sigma) \text{ for any } a_i^1 \in R_+,
\]

and
\[
  \pi_i^2(\sigma_i^2(h_i^1) \mid \sigma, h_i^1) \geq \pi_i^2(a_i^2 \mid \sigma, h_i^1) \text{ for any } a_i^2 \in R_+ \text{ and } h_i^1 \in R_+ \times Y.
\]

### 4 A Perfect Bayesian Equilibrium

We assume that the marginal cost of effort in each stage is increasing:

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\(^{11}\)Feedback policy \(f\) fails to be regular if \(f^{-1}(y)\) is, for example, the Cantor set for some \(y\).
**Assumption 1** For \( t = 1, 2 \), the cost function \( c_t : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies \( c'_t(0) = 0 \) and \( \inf_{a \in \mathbb{R}_+} c''_t(a) > 0 \).

We denote by \( P^{\sigma,f} \) the probability distribution induced by the pair of a regular feedback policy \( f \) and a strategy profile \( \sigma \), and by \( E^{\sigma,f} \) the corresponding expectation. The conditional expectation \( E^{\sigma,f}[\cdot \mid y] \) given the public announcement \( y \) is with respect to the conditional density \( g^{\sigma,f}_1(x_1 \mid y) \) of \( x_1 \) under the stage 1 effort profile \( \sigma_1 = (\sigma_1^1, \sigma_1^2) \). For \( x_1 \in f^{-1}(y) \), \( g^{\sigma,f}_1 \) can be explicitly written as

\[
g^{\sigma,f}_1(x_1 \mid y) = \frac{\phi_1(x_1 - \sigma_1^1 + \sigma_1^2)}{\int_{f^{-1}(y)} \phi_1(x_1' - \sigma_1^1 + \sigma_1^2) dx_1'} \quad \text{or} \quad \frac{\phi_1(x_1 - \sigma_1^1 + \sigma_1^2)}{\sum_{x_1' \in f^{-1}(y)} \phi_1(x_1' - \sigma_1^1 + \sigma_1^2)}
\]

depending on whether \( f^{-1}(y) \) has positive measure or is countable.\(^{12}\) Recall that the convolution of \( \phi_1 \) and \( \phi_2 \), denoted \( \phi_1 \ast \phi_2 \), is defined by

\[
(\phi_1 \ast \phi_2)(x) = \int_{\mathbb{R}} \phi_1(x-u) \phi_2(u) \, du.
\]

For any strategy \( \sigma^i = (\sigma_1^i, \sigma_2^i) \) of agent \( i \) and announcement \( y \in Y \), if we substitute agent \( i \)'s on-the-path stage 1 effort \( \sigma_1^i \) into his contingent stage 2 effort choice \( \sigma_2^i(\cdot, y) \), then \( \sigma_2^i(\sigma_1^i, y) \) equals the stage 2 effort along the path of play. For simplicity, we denote this quantity by \( \sigma_2^{i,0}(y) \). In other words, given \( \sigma^i, \sigma_2^{i,0}(y) \) is \( i \)'s stage 2 effort after announcement \( y \) when he makes no deviation. The following theorem characterizes the on-the-path effort levels in any pure PBE.

**Theorem 1** Suppose that Assumption 1 holds and that

\[
\sup_{x \in \mathbb{R}} |\phi'_2(x)| < \inf_{a \in \mathbb{R}_+} c''_2(a).
\]

If \( \sigma \) is a pure PBE under any feedback policy \( f \), then for any \( y \in Y \),

\[
\sigma_2^{1,0}(y) = \sigma_2^{2,0}(y) = \alpha_2(\sigma_1, y) \equiv (c'_2)^{-1} \left( E^{\sigma,f}[\phi_2(\tilde{x}_1) \mid y] \right).
\]

If, in addition, \( \sigma_1^1, \sigma_2^1 > 0 \), then

\[
\left\{ \begin{array}{l}
c'_1(\sigma_1^1) = (\phi_1 \ast \phi_2)(\sigma_1^1 - \sigma_2^1) + \int_{\mathbb{R}} c_2(\alpha_2(\sigma_1, f(x_1))) \phi_1(x_1 - \sigma_1^1 + \sigma_2^1) \, dx_1,

c'_1(\sigma_2^1) = (\phi_1 \ast \phi_2)(\sigma_1^1 - \sigma_2^1) - \int_{\mathbb{R}} c_2(\alpha_2(\sigma_1, f(x_1))) \phi_1(x_1 - \sigma_1^1 + \sigma_2^1) \, dx_1.
\end{array} \right.
\]

\(^{12}\)Note that \( g^{\sigma,f}_1 \) depends on \( \sigma \) only through the stage 1 profile \( \sigma_1 \). The explicit formulas of \( g^{\sigma,f}_1 \) are required for the description of the first-order conditions for an equilibrium.
Proof. See the Appendix. //

It should be noted that in any PBE, the agents’ stage 2 efforts are symmetric for any realization of the public announcement whether the equilibrium itself is symmetric or not. The stage 2 effort is determined through the standard marginal consideration: It balances the marginal increment in the probability of winning and the marginal cost of effort. (1) has the following implication. Suppose for simplicity that \( f \) is the full-feedback policy: \( f(x_1) = x_1 \). In this case, \( \sigma_{2,0}^1(x_1) = (c'_2)^{-1}(\phi_2(x_1)) \) as is readily verified. It follows that the stage 2 effort is the highest when \( x_1 \) is such that \( \phi_2(x_1) \) is the largest. When \( \phi_2 \) is unimodal at the origin as in the case of the normal distribution, hence, the stage 2 effort is a monotone decreasing function of \( |x_1| \). This corresponds to the standard intuition that the closer the competition, the higher the efforts the agents exert. Note, however, that this intuition fails when, for example, \( \phi_2 \) is bimodal so that \( \phi_2(x) = \phi_2(-x) > \phi_2(0) \) for some \( x > 0 \).

Let

\[
\epsilon = \frac{1}{2} \min \{ 1, \inf_{a \in \mathbb{R}_+} c'_1(a), \inf_{a \in \mathbb{R}_+} c'_2(a), \lim_{a \to \infty} c'_2(a) \} > 0.
\]

The next theorem identifies a sufficient condition for the existence of a pure PBE.

**Theorem 2** Suppose that Assumption 1 holds and that

\[
\sup_{x \in \mathbb{R}} \phi_2(x), \sup_{x \in \mathbb{R}} |\phi'_2(x)|, \int_{\mathbb{R}} |\phi''_2(x)|\, dx, \text{ and } \int_{\mathbb{R}} \frac{\phi'_2(x)^2}{\phi_1(x)}\, dx < \epsilon
\]

for \( \epsilon \) defined in (3). Given any feedback policy \( f \), there exists a pure PBE under \( f \) if (2) has a solution \( \sigma_1 = (\sigma_1^1, \sigma_1^2) \geq 0 \).

**Proof.** See the Appendix. //
noise is a standard requirement for the existence of an equilibrium in a tournament model with stochastic performance. Intuitively, if the noise is too small, then any infinitesimal increase in effort results in almost sure winning, making it impossible for the marginal equation to hold. It should also be emphasized that in Theorem 2, the noise level required for the existence of an equilibrium is independent of a particular feedback policy $f$.

In what follows, we assume for simplicity that $Y$ is a vector space and normalize $f(0) = 0 \in Y$. With this standardization, we say that a feedback policy $f$ is odd if $f(x) = -f(-x)$ for any $x \in R$ and even if $f(x) = f(-x)$ for any $x \in R$. Intuitively, if $f$ is odd, then the inference drawn from the announcement when agent $i$ leads agent $j$ in stage 1 is the exact opposite of that when their positions are reversed. On the other hand, if $f$ is even, then the announcement is the same regardless of the identity of the leader as long as the size of the lead is the same. For example, the full-feedback policy $f(x) = x$ is odd (but not even), whereas the no-feedback policy $f(x) \equiv 0$ is the only policy that is both odd and even.

A strategy profile $\sigma$ is symmetric if the two agents always choose the same effort level on the path: $\sigma_{1}^1 = \sigma_{2}^1$ and $\sigma_{1,0}^1(y) = \sigma_{2,0}^2(y)$ for any $y \in Y$. We now show that every even or odd policy admits a symmetric PBE when the noise is sufficiently large. By summing the two equations of (2), we see that the stage 1 effort in a symmetric PBE (if any) must satisfy

$$
\sigma_{1}^1 = \sigma_{1}^2 = a_{1}^* = (c_{1}^1)^{-1}(\phi_{1}^* \phi_{2}^*)(0)).
$$

The following theorem confirms that this $a_{1}^*$ is indeed the optimal choice in stage 1.

**Theorem 3** Suppose that Assumption 1 and condition (4) hold. If $f$ is either odd or even, there exists a unique symmetric pure PBE $\sigma$. Furthermore, for $\alpha_2$ defined in (1) and $a_{1}^*$ defined in (5), $\sigma$ satisfies

$$
\sigma_{1}^1 = \sigma_{1}^2 = a_{1}^*,
$$

and

$$
\sigma_{2,0}^1(y) = \sigma_{2,0}^2(y) = \alpha_2(\sigma_{1}, y) \text{ for any } y \in Y.
$$

**Proof.** See the Appendix. //
It should be noted that the stage 1 effort $a_1^*$ in the symmetric pure PBE is independent of the feedback policy $f$. Furthermore, the expected marginal cost in stage 2 equals the marginal cost in stage 1 since

$$E^\sigma,f[c'_2(\alpha_2(\sigma_1, \tilde{y}))] = E^\sigma,f[E^\sigma,f[\phi_2(\tilde{x}_1) \mid \tilde{y}]]$$

$$= E^\sigma,f[\phi_2(\tilde{x}_1)]$$

$$= (\phi_1 \ast \phi_2)(0)$$

$$= c'_1(a_1^*)$$

by the law of iterated expectation. Note, however, that we cannot use this intuitive equality as a basic premise for our analysis because of the presence of the strategic effect. The following facts about the no-feedback and full-feedback policies are immediate consequences of the above theorem.

**Proposition 4** Suppose that Assumption 1 holds. If $\sigma$ is the (unique) symmetric pure PBE under the no-feedback policy, then the stage 1 effort equals $\sigma^i_1 = a_1^*$ and the stage 2 effort equals

$$\sigma^i_{2,0} = a_2^* \equiv (c'_2)^{-1}(((\phi_1 \ast \phi_2)(0))) .$$

Likewise, if $\sigma$ is the (unique) symmetric pure PBE under the full-feedback policy, then the stage 1 effort equals $\sigma^i_1 = a_1^*$ and the expected stage 2 effort equals

$$E^\sigma,f[\sigma^i_{2,0}(\tilde{y})] = \int_{\mathbb{R}} (c'_2)^{-1}(\phi_2(x_1)) \phi_1(x_1) \, dx_1 .$$

When $(c'_2)^{-1}$ is concave or convex, Proposition 4 can be used to rank the no-feedback and full-feedback policies in terms of the expected stage 2 effort they induce in the symmetric PBE. Suppose for example that $(c'_2)^{-1}$ is concave. Then Jensen’s inequality implies that

Expected stage 2 effort under the full-feedback policy

$$= \int_{\mathbb{R}} (c'_2)^{-1}(\phi_2(x_1)) \phi_1(x_1) \, dx_1$$

$$\leq (c'_2)^{-1}\left(\int_{\mathbb{R}} \phi_2(x_1) \, dx_1\right) = a_2^*$$

$$= \text{Stage 2 effort under the no-feedback policy} .$$

The reverse inequality holds when $(c'_2)^{-1}$ is convex. The next section generalizes these comparisons to any feedback policies.
5 Optimal Feedback Policy

In this section, we will study the principal’s expected payoff in the pure PBE as identified in Theorems 1-3. We first make the following assumption about the principal’s payoff function:

**Assumption 2** When the stage 2 efforts satisfy $a_2 = a_2 = u$, $V(a_1, a_2)$ is a linear increasing function of $u$. In other words, there exist $A : \mathbb{R}_+ \to \mathbb{R}_+$ and $B : \mathbb{R}_+ \to \mathbb{R}$ such that if $a_1 = a_2 = u$, then $V(a_1, a_2) = A(a_1)u + B(a_1)$ for any $a_1$.

Given that the stage 2 efforts in any PBE are always symmetric between the two agents by (1), Assumption 2 ensures that the principal’s expected payoff is an increasing function of their expected stage 2 effort.\(^{14}\) The class of payoff functions $V$ which satisfy Assumption 2 includes $V(a) = \sum_t (a_1 + a_2)$, $V(a) = \sum_t \min \{a_1, a_2\}$, and more generally, the CES family $V(a) = \sum_t \{(a_1^m + (a_2^m)^m\}^{1/m}$ ($m \in \mathbb{R} \setminus \{0\}$). Let $v(\sigma, f)$ denote the principal’s expected payoff in a PBE $\sigma$ under the feedback policy $f$:

$$v(\sigma, f) = E^{\sigma,f} [V(\sigma_1, \sigma_2, 0(\tilde{y}))].$$

5.1 Symmetric Equilibrium

Even when $f$ admits multiple symmetric pure PBE’s, they all induce the same on-the-path effort by Theorem 1 and equation (5). In this sense, the principal’s payoff is independent of the choice of a symmetric PBE $\sigma$ and hence we define

$$\bar{v}^*(f) = \begin{cases} v(\sigma, f) & \text{if } f \text{ admits a symmetric pure PBE } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 5** Suppose that Assumptions 1, 2, and condition (4) hold. If the marginal cost function $c'_2$ for stage 2 is convex over $[0, (c'_2)^{-1}(\sup_{x \in \mathbb{R}} \phi_2(x))]$, then the no-feedback policy maximizes $\bar{v}^*$ among all policies.

**Proof.** Take any policy $f$ with a symmetric pure PBE $\sigma$. Since $(c'_2)^{-1}$ is concave over $[0, \sup_{x \in \mathbb{R}} \phi_2(x)]$, it follows from Jensen’s inequality and the law of iterated

\(^{14}\) In the consideration of a symmetric PBE in Section 4.1, we only need the linearity of $V(a_1, a_2 = a_2 = u)$ in $u$ for $a_1$ such that $a_1 = a_2^2$. 

15
expectation that the expected stage 2 effort level in equilibrium satisfies
\[
E^{\sigma,f} [\alpha_2(\sigma_1, \tilde{y})] = E^{\sigma,f} \left[ (c'_2)^{-1} \left( E^{\sigma,f} [\phi_2(\tilde{x}_1) \mid \tilde{y}] \right) \right]
\leq (c'_2)^{-1} \left( E^{\sigma,f} [\phi_2(\tilde{x}_1)] \right)
= (c'_2)^{-1} \left( ((\phi_1 \ast \phi_2)(0)) \right)
= a^*_2.
\]

Note that the last term \(a^*_2\) is the (expected) stage 2 effort under the no-feedback policy. Since the stage 1 effort in any symmetric PBE under any feedback policy is \(a^*_1\) by (5), and Assumption 2 implies that the principal’s expected payoff is an increasing function of the expected stage 2 effort, we conclude that the no-feedback policy yields the highest expected payoff to the principal in a symmetric PBE. //

**Theorem 6** Suppose that Assumptions 1, 2, and condition (4) hold. If the marginal cost function \(c'_2\) for stage 2 is concave over \([0, (c'_2)^{-1}(\sup_{x \in \mathbb{R}} \phi_2(x))]]\), then the full-feedback policy maximizes \(\bar{v}^*\) among all policies.

**Proof.** Take any policy \(f\) which admits a symmetric PBE \(\sigma\). By the same logic as in the proof of Theorem 5, it suffices to show that the expected stage 2 effort under \(f\) does not exceed that under the full-feedback policy. Since \((c'_2)^{-1}\) is convex over \([0, \sup_{x \in \mathbb{R}} \phi_2(x)]\), Jensen’s inequality now implies that
\[
E^{\sigma,f} [\alpha_2(\sigma_1, \tilde{y})] = E^{\sigma,f} \left[ (c'_2)^{-1} \left( E^{\sigma,f} [\phi_2(\tilde{x}_1) \mid \tilde{y}] \right) \right]
\leq E^{\sigma,f} \left[ (c'_2)^{-1} \left( (\phi_2(\tilde{x}_1)) \mid \tilde{y} \right) \right]
= E^{\sigma,f} [((\phi_1 \ast \phi_2)(0))].
\]
Since \(\sigma\) is symmetric, \(\sigma^*_1 = \sigma^*_2 = a^*_1\) by (5), and hence the last term of the above inequality equals
\[
a^*_2 = \int_{\mathbb{R}} (c'_2)^{-1}(\phi_2(x_1)) \phi_1(x_1) dx_1,
\]
which is the expected stage 2 effort in the symmetric PBE under the full-feedback policy. //

The proofs of Theorems 5 and 6 also indicate that when \(c'_2\) is concave (resp. convex), the no-feedback (resp. full-feedback) policy yields the lowest expected payoff
to the principal. On the other hand, when the marginal cost function $c'_2$ for stage 2 is linear (and hence both concave and convex), the induced effort in either stage is not affected by the feedback policy. The following corollary is an immediate consequence of Theorem 1.

**Corollary 7** Suppose that the stage 2 cost function is quadratic: $c_2(a) = \frac{1}{2} ka^2$ for some $k > 0$. Suppose also that Assumption 1 holds and that $\sup_{x \in \mathbb{R}} |\phi'_2(x)| < k$. Let $f$ be any feedback policy and $\sigma$ be a symmetric pure PBE under $f$. Then $\sigma_1^* = a_1^*$ and $\mathbb{E}^{\sigma,f}[\sigma^*_2(\tilde{y})] = \frac{1}{k} (\phi_1 * \phi_2)(0)$. It follows that the principal’s expected payoff $v(\sigma, f)$ is independent of $f$.

Consider now the feedback policy $f$ that reveals the size of the lead, but not the identity of the leader: $f(x_1) = |x_1|$ for every $x_1$. This policy is even and induces the same effort as the full-feedback policy in the two-stage model. To see this, note that

$$E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y} = f(x_1)] = \frac{\phi_2(x_1) + \phi_2(-x_1)}{2} = \phi_2(x_1)$$

by the symmetry of $\phi_2$. Hence, (1) implies that the stage 2 effort under $f$ equals that under the full-feedback policy. Since the stage 1 effort is the same under any policy in the symmetric PBE, it follows that $f$ is also an optimal policy under the conditions of Theorem 6. We will return to such a policy in the analysis of the $T$-stage model in Section 5.

### 5.2 Asymmetric Equilibrium

We now allow a PBE $\sigma$ to be asymmetric, and define

$$\bar{v}(f) = \sup \{ v(\sigma, f) : \sigma \text{ is a pure PBE under } f \text{ and satisfies (2)} \},$$

with $\bar{v}(f) = -\infty$ if the corresponding strategy profile does not exist. We will make some additional assumptions in order to evaluate the principal’s expected payoff in an asymmetric PBE. Specifically, we will identify the situations where the principal obtains a higher payoff in a symmetric PBE than in an asymmetric PBE. In such situations, the optimality of the no-feedback or full-feedback policies is obtained just as before.

**Assumption 3** $(\phi_1 * \phi_2)(0) = \max_{x \in \mathbb{R}} (\phi_1 * \phi_2)(x)$. 

17
It can be readily verified that Assumption 3 is implied by the unimodality (single-peakedness) of the densities $\phi_1$ and $\phi_2$: It holds, for example, when both $\phi_1$ and $\phi_2$ are normal distributions.

**Assumption 4** The principal’s payoff function $V$ is differentiable, and for any $a = (a^1_1, a^1_2), (a^2_1, a^2_2) \in \mathbb{R}^4_+$ such that $a^1_1 < a^2_1$ and $a^1_2 = a^2_2$, we have

$$
\frac{c''_1(a^1_1) - 2(\phi_1 * \phi_2)'(a^1_1 - a^2_1)}{c''_1(a^2_1) + 2(\phi_1 * \phi_2)'(a^1_1 - a^2_1)} < \frac{\partial V}{\partial a^1_1}(a)
$$

Intuitively, this assumption requires that the two agents’ efforts be complementary to each other from the point of view of the principal. In other words, a very high effort from one agent is of little value to the principal if the other agent’s effort is low: The principal would rather have both agents make moderate efforts. Formally, the left-hand side of (6) represents the slope of the curve

$$
h(a^1_1, a^2_1) \equiv c'_1(a^1_1) + c'_1(a^2_1) - 2(\phi_1 * \phi_2)(a^1_1 - a^2_1) = 0
$$

in the $(a^1_1, a^2_1)$-plane, which equals the sum of the two first-order conditions in (2). On the other hand, the right-hand side of (6) represents the slope of the principal’s iso-payoff curve. Hence, (6) is a single-crossing condition asserting that the iso-payoff curve always has a steeper slope than (7). To see that this implies complementarity between the two agents’ efforts, suppose that $V$ has the CES form:

$$
V(a) = \sum_t \{ (a^1_t)^m + (a^2_t)^m \}^{1/m} (m \in \mathbb{R} \setminus \{0\}).
$$

In this case, the right-hand side of (6) equals $(a^1_1/a^2_1)^{m-1}$. Hence, (6) is easy to satisfy when $m - 1$ is negative and large in absolute value. In particular, it will hold for any $c_1$ as $m \to -\infty$, or $V(a) = \sum_t \min \{a^1_t, a^2_t\}$ in the limit. On the other hand, the inequality fails if $m > 1$ and $c'_1$ is concave.

As seen in the Appendix (Lemma 17), Assumptions 3 and 4 together guarantee that along (7), the principal’s payoff is maximized at the symmetric point $(a^*_1, a^*_1)$ (provided that the stage 2 efforts are symmetric). The next theorem identifies when the no-feedback policy is optimal under these conditions.

**Theorem 8** Suppose that Assumptions 1-4 and condition (4) hold. If the marginal cost function $c'_2$ for stage 2 is convex over $[0, (c'_2)^{-1}(\sup_{x \in \mathbb{R}} \phi_2(x))])$, then the no-feedback policy maximizes $\bar{v}(\cdot)$ among all policies.
Proof. See the Appendix. //

For the other type of the conclusion, we need the density function of the stage 2 noise to have a single peak as in the case of the normal distribution.

**Assumption 5** $\phi_2$ is unimodal at 0: $\phi_2$ is strictly increasing over $(-\infty, 0)$ and strictly decreasing over $(0, \infty)$.

Clearly, $\sup_{x \in \mathbb{R}} \phi_2(x) = \phi_2(0)$ under Assumption 5.

**Theorem 9** Suppose that Assumptions 1-5 and condition (4) hold. If the marginal cost function $c_2'$ for stage 2 is concave over $[0, (c_2')^{-1}(\phi_2(0))]$, then the full-feedback policy maximizes $\bar{v}(\cdot)$ among all policies.

Proof. See the Appendix. //

As in Section 4.1, we can show that the principal’s payoff is independent of the choice of a feedback policy when the stage-cost function $c_2$ is quadratic. It can also be verified as before that the feedback policy that reveals the absolute value of the score is optimal under the conditions of Theorem 9.

### 5.3 Optimality of Intermediate Policies

We now turn to the question of optimal policies when the marginal cost of effort is neither convex nor concave over the relevant domain. Given the conclusions of the preceding sections, we are specifically interested in the existence of a feedback policy that induces a higher effort than the no-feedback or full-feedback policies in such a circumstance. For simplicity, we suppose that $\phi_2$ is unimodal at 0 (Assumption 5), and that the stage 2 marginal cost function $c_2'$ has a single reflection point $r$ in the range $[0, (c_2')^{-1}(\phi_2(0))]$. More specifically,

**Assumption 6** There exists $r \in (0, (c_2')^{-1}(\phi_2(0)))$ such that the stage 2 marginal cost function $c_2'$ is convex over $[0, r]$ and concave over $[r, (c_2')^{-1}(\phi_2(0))]$.

Figure 1 illustrates $c_2$ satisfying the above condition. Let $\rho > 0$ be such that $\phi_2(\rho) = \phi_2'(r)$. It follows that when the stage 1 score is such that $|x_1| < \rho$, then $\phi_2(x_1) > \phi_2'(r)$ so that $c_2'$ is concave at $(c_2')^{-1}(\phi_2(x_1))$, and likewise, when $|x_1| \geq \rho$, then $\phi_2(x_1) \leq \phi_2'(r)$ so that $c_2'$ is convex at $(c_2')^{-1}(\phi_2(x_1))$. By the discussion in the previous sections, a natural focus is on the feedback policy that reveals full...
Recall that \( \bar{\rho} \) of (9) equals the expected stage 2 effort under any \( f \). It follows that the expected stage 2 effort under \( f^\ast \)

\[
f^\ast(x_1) = \begin{cases} x_1 & \text{if } |x_1| < \rho, \\ \rho & \text{otherwise}. \end{cases}
\]

Note that the announcement \( \rho \) under \( f^\ast \) merely indicates the fact that \( |x_1| \geq \rho \). Recall that \( \bar{\nu}^\ast(f) \) denotes the principal’s payoff in the symmetric PBE under \( f \).

**Theorem 10** Suppose that Assumptions 1, 2, 5, 6 and condition (4) hold. Consider a class of feedback policies \( f \) which reveal whether \( |x_1| < \rho \) or not, i.e., \( f(x_1) \neq f(x_1') \) for any \( x_1 \) and \( x_1' \) such that \( |x_1| < \rho \) and \( |x_1'| \geq \rho \). Then \( f^\ast \) specified in (8) maximizes \( \bar{\nu}^\ast \) in this class.

**Proof.** Let \( \sigma \) be the symmetric PBE under any feedback policy \( f \) in such a class. We then have

\[
\begin{align*}
|x_1| < \rho & \implies E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y} = f(x_1)] > c_2'(r), \quad \text{and} \\
|x_1| \geq \rho & \implies E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y} = f(x_1)] \leq c_2(r)
\end{align*}
\]

It follows that the expected stage 2 effort under \( f \) satisfies

\[
\begin{align*}
E^{\sigma,f]\left( (c_2')^{-1} \left( E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y}] \right) \right) & \\
= E^{\sigma,f}\left[ (c_2')^{-1} \left( E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y}] \right) \right] | |\tilde{x}_1| \geq \rho] P^{\sigma,f}(|\tilde{x}_1| \geq \rho) \\
& + E^{\sigma,f}\left[ (c_2')^{-1} \left( E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y}] \right) \right] | |\tilde{x}_1| < \rho] P^{\sigma,f}(|\tilde{x}_1| < \rho) \\
\leq (c_2')^{-1} \left( E^{\sigma,f}[\phi_2(\tilde{x}_1) | \tilde{y}] \right) | |\tilde{x}_1| \geq \rho] P^{\sigma,f}(|\tilde{x}_1| \geq \rho) \\
& + E^{\sigma,f}\left[ (c_2')^{-1} \left( \phi_2(\tilde{x}_1) \right) \right] | |\tilde{x}_1| < \rho] P^{\sigma,f}(|\tilde{x}_1| < \rho) \\
& = (c_2')^{-1} \left( E^{\sigma,f}[\phi_2(\tilde{x}_1) | |\tilde{x}_1| \geq \rho] \right) P^{\sigma,f}(|\tilde{x}_1| \geq \rho) \\
& + E^{\sigma,f}\left[ (c_2')^{-1}(\phi_2(\tilde{x}_1)) \right] | |\tilde{x}_1| < \rho] P^{\sigma,f}(|\tilde{x}_1| < \rho),
\end{align*}
\]

where the inequality follows from the above observation as well as Jensen’s inequality, and the last equality from the fact that the filtration induced by the announcement \( \tilde{y} \) includes the events \( \{ |\tilde{x}_1| \geq \rho \} \) and \( \{ |\tilde{x}_1| < \rho \} \). Since the far right-hand side of (9) equals the expected stage 2 effort under \( f^\ast \), the desired conclusion follows. //

The above theorem readily implies:
Corollary 11 Suppose that Assumptions 1, 2, 5, 6 and condition (4) hold. Then $f^*$ specified in (8) yields the principal a higher expected payoff than the full feedback policy.

The proof of Theorem 10 also suggests that any policy $f$ measurable with respect to some subset of $\{x_1 : |x_1| < \rho\}$ or $\{x_1 : |x_1| \geq \rho\}$ is dominated by another policy. To be more precise, let $B \subset \{x_1 : |x_1| < \rho\}$ and suppose that $f$ is measurable with respect to $B$, i.e., $f(x_1) \neq f(x'_1)$ for any $x_1$ and $x'_1$ such that $x_1 \in B$ and $x'_1 \notin B$. Then a slight modification of the above proof shows that $f$ is (weakly) dominated by an alternative policy that reveals full information when $x_1 \in B$, but is the same as $f$ when $x_1 \notin B$. The following theorem extends this kind of logic further to give a sufficient condition for the no-feedback policy to be suboptimal.

Theorem 12 Suppose that Assumptions 1, 2, 5, and 6 and condition (4) hold. If $(\phi_1 \phi_2)(0) > c'_2(r)$,

then there exists a feedback policy that induces a higher expected effort than the no-feedback policy.

Proof. See the Appendix. //

The proof of the above theorem consists of showing that the no-feedback policy is dominated by a policy which only reveals whether or not $\phi_2(x_1)$ has exceeded a certain threshold.

6 $T$-Stage Tournament

Suppose now that the tournament is over $T$ stages: In each stage, the two agents choose effort levels, the performance score is realized, and the principal makes an announcement. It should be noted that a feedback policy in this generalized setting is significantly more complex than that in the two-stage model. Note specifically that the principal’s announcement after stage $t$ may depend not only on the performance score in stage $t$ but also on those in stages $1, \ldots, t - 1$. In other words, a feedback policy controls not only the amount of information revelation but also its timing. For example, the principal may wish to withhold some information for some time but may release it with a lag conditional on subsequent developments.
Extending the horizon of the tournament is accompanied by some technical complications. In particular, a symmetric PBE in the $T$-stage model for $T \geq 3$ exists only if a feedback policy does not reveal the identity of the leader.\footnote{We also need stronger conditions on the distributions of noise. The bulk of the proof in the Appendix is devoted to the existence of an equilibrium.} This implies that the full-feedback policy is excluded from our analysis. Despite this, however, it can be shown that the qualitative conclusions from the two-stage model continue to hold. Specifically, it is shown that revealing no information is optimal when the marginal stage cost functions are concave, and the opposite is true when they are convex.

Formally, let $a_i^t$ denote agent $i$'s effort in stage $t$. The score $x_t$ in stage $t$ equals $a_1^t - a_2^t + \zeta_t$, where $\zeta_t$ is a random variable with the strictly positive density $\phi_t$ over $\mathbb{R}$. We assume that $\phi_t$ is symmetric around zero, and twice continuously differentiable. For each $t = 1, \ldots, T$, denote by $\omega_t = (x_1, \ldots, x_t)$ the sequence of scores in stages $1, \ldots, t$, and by $\Delta_t$ the aggregate score at the end of stage $t$:

$$\Delta_t = \sum_{s=1}^{t} x_s.$$ 

Given $\omega_t$ and $s < t$, we also use $\omega_s$ to denote the $s$-length truncation of $\omega_t$.

The feedback policy in the $T$-stage tournament is a pair $(f, Y)$, where $Y$ is a collection of sets $Y_1, \ldots, Y_{T-1}$ and $f$ is a collection of (measurable) mappings $f_1, \ldots, f_{T-1}$ such that

$$f_t : \mathbb{R}^t \rightarrow Y_t \quad \text{for } t = 1, \ldots, T - 1.$$ 

The interpretation is that $Y_t$ is the set of possible announcements after stage $t$. As noted above, this definition allows the announcement $y_t = f_t(\omega_t)$ after stage $t$ to depend on all past scores and not just the stage $t$ score. As before, we call the mappings $f$ itself a feedback policy and omit explicit reference to the sets $Y$. As an example of a feedback policy that does not have a counterpart in the two-stage model, let $(f, Y)$ be such that $Y_1 = \{0\}$, $Y_2 = \cdots = Y_{T-1} = \{0, 1\}$, $f_1(x_1) = 0$ for any $x_1$, and

$$f_t(x_1, \ldots, x_t) = \begin{cases} 1 & \text{if } |x_1 + \cdots + x_t| < |x_1 + \cdots + x_{t-1}|, \\ 0 & \text{otherwise}, \end{cases}$$

for $t = 2, \ldots, T - 1$. In other words, this feedback policy announces after each stage whether or not the lead has widened.
Given any feedback policy \( f \), let 
\[
Z_t(f) = \{ z_t = (y_1, \ldots, y_t) : y_s = f_s(\omega_s) \text{ for } s = 1, \ldots, t \text{ for some } \omega_t \}
\]
be the set of sequences of possible announcements after stages 1, \ldots, \( t \), and for each \( z_t \in Z_t(f) \), let 
\[
X^f_t(z_t) = \{ \omega_t : f_s(\omega_s) = y_s \text{ for } s = 1, \ldots, t \}
\]
be the set of scores compatible with \( z_t \). In the \( T \)-period model, we say that \( f \) is regular if \( X^f_t(z_t) \) has positive measure or is countable for every \( z_t \) and \( t \). The analysis in what follows assumes regular policies.

In order to avoid technical complications arising from boundary problems, we assume that efforts can take negative values.

**Assumption 7** For each \( t = 1, \ldots, T \), the cost function \( c_t \) in stage \( t \) is a mapping from \( \mathbb{R} \) to \( \mathbb{R}_+ \) and satisfies \( c_t(0) = c'_t(0) = 0 \), and \( \inf_{a \in \mathbb{R}} c''_t(a) > 0 \).

Under Assumption 7, the cost of negative effort is positive and increases (at an increasing rate) with its absolute size. As seen below, however, only positive efforts are observed along any symmetric equilibrium path.

Agent \( i \)'s history after stage \( t \), denoted \( h^i_t \), is the sequence of his effort choices \( b^i_t \equiv (a^i_1, \ldots, a^i_t) \) in stages 1, \ldots, \( t \) along with the sequence of public announcements \( z_t = (y_1, \ldots, y_t) \) after stages 1, \ldots, \( t \). Agent \( i \)'s strategy \( \sigma^i \) is a sequence \( (\sigma_1, \ldots, \sigma_T) \), where \( \sigma_t : \mathbb{R}^t - 1 \times Z_{t-1}(f) \to \mathbb{R} \) specifies the effort level in stage \( t \) as a function of his history \( h^i_{t-1} \) after stage \( t - 1 \).

Given any feedback policy and a strategy profile \( \sigma \), let \( \pi^i_1(a^i_1 \mid \sigma) \) be agent \( i \)'s expected payoff in the entire game when he chooses \( a^i_1 \) in stage 1 and then plays according to \( \sigma^i \) in stages 2, \ldots, \( T \), and agent \( j \) plays according to \( \sigma^j \) in stages 1, \ldots, \( T \). Likewise, let \( \pi^i_t(a^i_t \mid \sigma, h^i_{t-1}) \) be agent \( i \)'s expected payoff over stages \( t, \ldots, T \) (i.e., utility from the possible prize minus the cost of efforts in stages \( t, \ldots, T \)) when he chooses \( a^i_t \) in stage \( t \) and plays according to \( \sigma^i \) in stages \( t+1, \ldots, T \), his history equals \( h^i_{t-1} \), and agent \( j \) plays according to \( \sigma^j \). A strategy profile \( \sigma \) is a perfect Bayesian equilibrium (PBE) under policy \( f \) if for \( i = 1, 2 \),
\[
\pi^i_1(\sigma^i_1 \mid \sigma) \geq \pi^i_1(a^i_1 \mid \sigma) \text{ for every } a^i_1 \in \mathbb{R}, \text{ and}
\]
\[
\pi^i_t(\sigma^i_t(h^i_{t-1}) \mid \sigma, h^i_{t-1}) \geq \pi^i_t(a^i_t \mid \sigma, h^i_{t-1}) \text{ for every } a^i_t \in \mathbb{R}, h^i_{t-1} \in \mathbb{R}^{t-1} \times Z_{t-1}(f), \text{ and } t = 2, \ldots, T.
\]
As before, denote by $\sigma^i_{t,0}(z_{t-1})$ agent $i$’s effort level in stage $t$ induced by $\sigma^i_t$ along the sequence of public announcements $z_{t-1} = (y_1, y_2, \ldots, y_{t-1})$: $\sigma^i_{1,0}(z_0) = \sigma^i_1$, $\sigma^i_{2,0}(z_1) = \sigma^i_2(\sigma^i_{1,0}(z_0), y_1)$, $\sigma^i_{3,0}(z_2) = \sigma^i_3(\sigma^i_{2,0}(z_1), y_2)$, and so on. A PBE is symmetric if the effort levels are symmetric on the path, i.e., $\sigma^i_{t,0}(z_{t-1}) = \sigma^j_{t,0}(z_{t-1})$ for any $z_{t-1} \in Z_{t-1}(f)$ and $t = 1, \ldots, T$.

We say that the feedback policy $f$ is even if $f_t(\omega_t) = f_t(-\omega_t)$ for every $\omega_t$ and $t$. Clearly, the no-feedback policy is even, but the full-feedback policy is not. In short, an even policy makes the same announcement when the two agents’ positions are reversed in all the past stages. As mentioned earlier, only even policies admit a symmetric PBE when $T \geq 3$. The following theorem provides sufficient conditions for the existence of a symmetric PBE under any even feedback policy in the $T$-stage model. It also characterizes the effort levels on the equilibrium path.

**Theorem 13** Suppose that Assumption 7 holds. Then there exists $\epsilon > 0$ such that if

\begin{equation}
(10) \quad \phi_T(x), |\phi_T'(x)| < \epsilon \text{ for any } x \in \mathbb{R}, \text{ and}
\end{equation}

\begin{equation*}
|\phi_t'(x)|, |\phi_t''(x)|, \left| \frac{\phi_t''(x)}{\phi_t'(x)} \right|, \left| \frac{\phi_t'''(x)}{\phi_t''(x)} \right| < \epsilon \quad \text{for any } x \in \mathbb{R} \text{ and } t = 1, \ldots, T - 1,
\end{equation*}

then for any even feedback policy $f$, there exists a unique symmetric pure PBE $\sigma = (\sigma_1, \ldots, \sigma_T)$ of the $T$-stage tournament. Furthermore, the stage $t$ effort on the equilibrium path in the symmetric PBE is given by

\begin{equation}
(11) \quad \sigma^i_{t,0}(z_{t-1}) = (c^i_t)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\tilde{\Delta}_{T-1}) \big| z_{t-1} \right] \right)
\end{equation}

for any $z_{t-1} \in Z_{t-1}(f)$ and $t = 1, \ldots, T$.

**Proof.** See the Appendix. //

When $X^f_{t-1}(z_{t-1})$ has positive measure, for example, (11) can be rewritten more explicitly as

\begin{equation*}
\sigma^i_{t,0}(z_{t-1}) = (c^i_t)^{-1} \left( \int_{X^f_{t-1}(z_{t-1})} \phi_T(\Delta_{T-1}) \prod_{s=t}^{T-1} \phi_s(x_s) g^f_{t-1}(\omega_{t-1} \mid z_{t-1}) d\omega_{T-1} \right),
\end{equation*}

where

\begin{equation*}
g^f_{t-1}(\omega_{t-1} \mid z_{t-1}) = \begin{cases} 
\prod_{s=t}^{T-1} \frac{\phi_s(x_s)}{\prod_{s=t}^{T-1} \phi_s(x'_s)} d\omega_{t-1} & \text{if } \omega_{t-1} \in X^f_{t-1}(z_{t-1}), \\
0 & \text{otherwise}
\end{cases}
\end{equation*}

\text{for any } z_{t-1} \in Z_{t-1}(f) \text{ and } t = 1, \ldots, T.
is the density of $\omega_{t-1} = (x_1, \ldots, x_{t-1})$ conditional on the sequence of public announcements $z_{t-1} = (y_1, \ldots, y_{t-1})$ under a symmetric strategy profile.\footnote{Note that $g_{t-1}^f(\omega_{t-1} \mid z_{t-1})$ is independent of any particular symmetric strategy profile $\sigma$. The similar expression applies when $X_{t-1}^f(z_{t-1})$ is countable.}

It can be seen that the symmetric PBE of Theorem 13 is a direct extension of that in Theorems 2 and 3 on the two-stage model. As in the two-stage model (Theorem 2), the existence of an equilibrium requires sufficiently high noise through condition (10), where $\epsilon$ is taken independent of a feedback policy.\footnote{Note that (10) for $T = 2$ is more restrictive than (4). For example, the normal distribution with a large variance satisfies (4) but not (10). On the other hand, (10) allows a version of the exponential distribution $f(x) = \gamma e^{-\gamma|x|}$ ($\gamma > 0$). The stronger conditions simplify the proof of the existence of a PBE in the $T$-stage setting.}

We now study how the choice of a feedback policy affects the expected stage effort in a symmetric PBE. Note from (11) that under any feedback policy, the stage 1 effort in a symmetric PBE equals

$$\sigma^i_{1,0} = (c'_1)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\tilde{\Delta}_{T-1}) \right] \right) = (c'_1)^{-1} \left( (\phi_1 \ast \cdots \ast \phi_T)(0) \right).$$

As was true with the two-stage model, hence, the stage 1 effort in a symmetric PBE is independent of a feedback policy. On the other hand, the expected efforts in stages 2, $\ldots$, $T$ do depend on the choice of a feedback policy. Consider first the no-feedback policy. In this case, the posterior belief over histories of performance scores in stage $t$ is the same as the prior and given by $g_{t-1}^f(\omega_{t-1} \mid z_{t-1}) = \prod_{s=1}^{t-1} \phi_s(x_s)$. Denote by $a_t^*$ the (deterministic) stage $t$ effort in a symmetric PBE under the no-feedback policy. Then Theorem 13 shows that $a_t^*$ is given by

$$a_t^* = (c'_t)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\tilde{\Delta}_{T-1}) \right] \right)$$

$$= (c'_t)^{-1} \left( \int_{R^{T-1}} \phi_T(\Delta_{T-1}) \prod_{s=1}^{T-1} \phi_s(x_s) \, dx_1 \cdots dx_{T-1} \right)$$

$$= (c'_t)^{-1} \left( (\phi_1 \ast \cdots \ast \phi_T)(0) \right),$$

where $\phi_1 \ast \cdots \ast \phi_T$ is the convolution of $\phi_1, \ldots, \phi_T$. Consider next an even policy $f$ which reveals the absolute value of the aggregate score at the end of each stage:

$$f_t(\omega_t) = |\Delta_t| \quad t = 1, \ldots, T - 1.$$ 

As noted earlier, this policy induces the same level of effort as the full-feedback policy in the two-stage model. Let $\sigma$ be the symmetric pure PBE under $f$. By
symmetry, we have

\[ P^\sigma_f(\tilde{\Delta}_{t-1} = y_{t-1} \mid z_{t-1}) = P^\sigma_f(\tilde{\Delta}_{t-1} = -y_{t-1} \mid z_{t-1}) = \frac{1}{2}. \]

It hence follows from (11) that

\[
c'_t(\sigma^i_t(z_{t-1})) = \int_{R^{T-t}} \left\{ \frac{1}{2} \phi_T \left( y_{t-1} + \sum_{s=t}^{T-1} x_s \right) + \frac{1}{2} \phi_T \left( -y_{t-1} + \sum_{s=t}^{T-1} x_s \right) \right\} \
\times \prod_{s=t}^{T-1} \phi_s(x_s) \, dx_t \cdots dx_{T-1} 
= \frac{1}{2} (\phi_t * \cdots * \phi_T)(-y_{t-1}) + \frac{1}{2} (\phi_t * \cdots * \phi_T)(y_{t-1}) 
= (\phi_t * \cdots * \phi_T)(y_{t-1}).
\]

Therefore, the expected stage \( t \) effort under \( f \) equals

\[ E^{\sigma,f}[\sigma^i_t(\tilde{z}_{t-1})] = \int_{R^{T-t}} (c'_t)^{-1} ((\phi_t * \cdots * \phi_T)(\Delta_{t-1})) \prod_{s=t}^{T-1} \phi_s(x_s) \, d\omega_{t-1}. \]

The following theorems claim the optimality of each one of the above two policies depending on whether the marginal stage cost function is convex or concave.

**Theorem 14** Suppose that Assumption 7 holds and that condition (10) holds for a sufficiently small \( \epsilon \). For any \( t = 2, \ldots, T \), if the stage \( t \) marginal cost \( c'_t \) is convex, then the no-feedback policy \( f \) maximizes the expected stage \( t \) effort in the symmetric PBE among all even policies.

**Proof.** Let \( f \) be any symmetric PBE that admits a symmetric PBE \( \sigma \). Since \( (c'_t)^{-1} \) is concave, it follows from Jensen’s inequality that agent \( i \)'s stage \( t \) effort in (11) under \( \sigma^i \) satisfies

\[
E^{\sigma,f}[\sigma^i_t(\tilde{z}_{t-1})] = E^{\sigma,f}[(c'_t)^{-1} \left( E^{\sigma,f}[\phi_T(\tilde{\Delta}_{T-1}) \mid \tilde{z}_{t-1}] \right)] 
\leq (c'_t)^{-1} \left( E^{\sigma,f} \left[ E^{\sigma,f}[\phi_T(\tilde{\Delta}_{T-1})] \mid \tilde{z}_{t-1}] \right] \right) 
= (c'_t)^{-1} \left( E^{\sigma,f}[\phi_T(\tilde{\Delta}_{T-1})] \right) 
= (c'_t)^{-1} ((\phi_1 * \cdots * \phi_T)(0)) = a^*_t,
\]

where \( a^*_t \) is the stage \( t \) effort under the no-feedback policy as specified above. //
Theorem 15 Suppose that Assumption 7 holds and that condition (10) holds for a sufficiently small $\epsilon > 0$. For any $t = 2, \ldots, T$, if the stage $t$ marginal cost $c'_t$ is concave, then the feedback policy in (12) that reveals the absolute value of the aggregate score after every stage maximizes the expected stage $t$ effort in the symmetric PBE among all even policies.

Proof. See the Appendix. //

The above theorems translate into those on the principal’s payoffs if his payoff function $V : R^{2T} \rightarrow R$ is increasing in the agents’ expected effort in each stage. Formally, assume that $V$ is increasing ($V(\hat{a}) \geq V(a)$ if $\hat{a} \geq a$), symmetric ($V(a) = V(\hat{a})$ if $a_1 = \hat{a}_2$ and $a_2 = \hat{a}_1$), and satisfies Assumption 8 below.

Assumption 8 When the two agents’ efforts satisfy $a_1^t = a_2^t = u_t$ for $t = 1, \ldots, T$, $V(a_1, a_2, \ldots, a_T)$ is linear in $(u_2, \ldots, u_T)$. In other words, there exist $A_t : R_+ \rightarrow R_+ (t = 2, \ldots, T)$ and $B_t : R_+ \rightarrow R$ such that if $a_1^t = a_2^t = u_t$ for $t = 1, \ldots, T$, then

$$V(a_1, a_2, \ldots, a_T) = \sum_{t=2}^{T} A_t(u_1) u_t + B_t(u_1).$$

As in Section 4, let $v(\sigma, f) = E^{\sigma,f} \left[V(\sigma_1, \sigma_2, 0(\hat{z}_1), \ldots, \sigma_T, 0(\hat{z}_{T-1}))\right]$, and

$$\bar{v}^*(f) = \sup \{v(\sigma, f) : \sigma \text{ is a symmetric PBE under } f\}.$$  

The following corollary is then an immediate consequence of Theorems 14 and 15.

Corollary 16 Suppose that Assumptions 7 and 8 hold and that condition (10) holds for a sufficiently small $\epsilon > 0$.

(i) If marginal stage cost function $c'_t$ is convex for $t = 2, \ldots, T$, then the no-feedback policy maximizes $\bar{v}^*$ among all even policies.

(ii) If marginal stage cost function $c'_t$ is concave for $t = 2, \ldots, T$, then the feedback policy in (12) maximizes $\bar{v}^*$ among all even policies.

7 Conclusion

The paper gives a first attempt to understand the use of the designer’s private information in a dynamic tournament, and its conclusion shows that the principal’s
optimal feedback policy depends sensitively on the functional form of the agents’ disutility of effort. Although the present model abstracts from many important features of real tournaments, we believe that such sensitivity to parameters is at the core of the information revelation problem. In this sense, our findings suggest that we need to proceed with caution when generalizing conclusions obtained from particular parametrizations.

One important property of a PBE used extensively in our analysis is the symmetry of the stage 2 efforts between the agents. This property fails in some important cases. The first is when we consider a mixed equilibrium. In a mixed PBE where an agent’s stage 1 effort is randomly chosen, his stage 2 effort is naturally a function of the realized stage 1 effort. This implies that the two agents’ stage 2 efforts are almost always different. Second, in some applications, it may be more appropriate to suppose that the principal can privately send a personalized message to each agent. For example, Mares and Harstad (2002) show in a common-value auction setting that an auctioneer may be better off revealing his private information in a non-public way. Such might as well be the case in the present model. However, private feedback of information will again yield an asymmetry in the agents’ stage 2 efforts. In both cases, asymmetry makes no explicit characterization of the stage 2 effort possible.

Another key assumption of our model is that the principal commits to his feedback policy for any realization of his private signal. It should be noted that such commitment is a standard assumption in the mechanism design literature. A totally different conclusion is possible under an alternative hypothesis that the principal lacks commitment power and optimizes his announcement after seeing his signal. One interpretation of the principal’s commitment to his feedback policy is through enforcement by a third party. In other words, we may suppose that a third party monitor punishes the principal for any deviation from his policy. When we take such an interpretation, however, it is important to note that not all policies are equally credible. For example, suppose that the principal declares the use of the no-feedback policy. In this case, any release of information afterward is a clear indication of his deviation. In this sense, the principal would find it difficult to deviate from his policy. On the other hand, if the principal announces that he will use

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18 In the common analysis of auctions, for example, an auctioneer retains his good if no bid reaches the reserve price.

19 Kaplan and Zamir (2000) find that the auctioneer cannot exploit his private information on the bidders’ valuation if he cannot commit to an announcement policy.
the full-feedback policy, his deviation cannot be detected unless his announcement is compared against his private information. This suggests that the principal may find more leeway to cover up his deviation. Such a variation in credibility levels would be an important issue when taking the enforcement interpretation. Alternatively, even when the private information is not verifiable, it may still be possible to use statistical testing to enforce a feedback policy when the tournaments are repeated over time under the same policy. For example, a statistical test would reject the trustworthiness of a tournament organizer who always reports a close race for the sake of spurring competition. The analysis of such a model, however, is not straightforward.

These are left as topics of future research.

Appendix

Recall that the set $f^{-1}(y)$ or $X_{t-1}^f(z_{t-1})$ of scores compatible with the announcement $y$ or $z_{t-1}$ either has positive measure or is countable by the regularity assumption. When the distinction is necessary, equations in the Appendix assume for simplicity that the set $f^{-1}(y)$ or $X_{t-1}^f(z_{t-1})$ has positive measure. When it is countable, the same argument goes through by replacing any integral with respect to the conditional density by the summation over $f^{-1}(y)$ or $X_{t-1}^f(z_{t-1})$.

Proof of Theorem 1 Fix any PBE $\sigma$. With slight abuse of notation, let $g_{\sigma^f}^i(x_1 \mid a_1^i, y)$ denote the density of $x_1$ conditional on the public announcement $y$ when the stage 1 actions are $a_1^i$ for agent $i$ and $\sigma_1^j$ for agent $j$: For example, when $f^{-1}(y)$ has positive measure, we have for $x_1 \in f^{-1}(y)$,

$$g_{\sigma^f}^i(x_1 \mid a_1^i, y) = \frac{\phi_1(x_1 - a_1^i + \sigma_1^2)}{\int_{f^{-1}(y)} \phi_1(x_1' - a_1^i + \sigma_1^2) \, dx_1'}, \quad \text{and}$$

$$g_{\sigma^f}^i(x_1 \mid a_2^i, y) = \frac{\phi_1(x_1 - \sigma_1^2 + a_2^i)}{\int_{f^{-1}(y)} \phi_1(x_1' - \sigma_1^2 + a_2^i) \, dx_1'}.$$

Note in particular that $g_{\sigma^f}^i(x_1 \mid \sigma_1^1, y) = g_{\sigma^f}^i(x_1 \mid \sigma_1^2, y) = g_{\sigma^f}^i(x_1 \mid y).$ Recall that $\pi_2^i(a_2^i \mid \sigma, a_1^i, y)$ represents agent $i$’s expected payoff in stage 2 when he chooses $a_2^i$ in stage 2, his history after stage 1 is $h_1^i = (a_1^i, y)$, and agent $j$ plays according to the equilibrium strategy $\sigma^j$. For simplicity, write $\pi_2^i(a_2^i \mid a_1^i, y)$ for $\pi_2^i(a_2^i \mid \sigma, a_1^i, y)$.
It can be seen that \( \pi^i_2(a^i_2 \mid a^i_1, y) \) is written as

\[
\pi^i_2(a^i_2 \mid a^i_1, y) = \int_R \Phi_2(a^i_2 - \sigma^2_{2,0}(y) + x_1) g^i_1 f(x_1 \mid a^i_1, y) \, dx_1 - c_2(a^i_2)
\]

for agent 1, and

\[
\pi^i_2(a^i_2 \mid a^i_2, y) = \int_R \Phi_2(-\sigma^2_{2,0}(y) + a^i_2 - x_1) g^i_2 f(x_1 \mid a^i_2, y) \, dx_1 - c_2(a^i_2)
\]

for agent 2. Differentiating \( \pi^i_2 \) with respect to \( a^i_2 \), we obtain

\[
\frac{\partial \pi^i_2}{\partial a^i_2}(a^i_2 \mid a^i_1, y) = \int_R \phi_2(a^i_2 - \sigma^2_{2,0}(y) + \tilde{x}_1) g^i_1 f(x_1 \mid a^i_1, y) \, dx_1 - c_2'(a^i_2).
\]

Since \( c'_2(0) = 0 \) implies \( \frac{\partial \pi^i_2}{\partial a^i_2}(0 \mid a^i_1, y) > 0 \), the equilibrium action \( \sigma^i_2(a^i_1, y) \) (if any) must satisfy

\[
\frac{\partial \pi^i_2}{\partial a^i_2}(\sigma^i_2(a^i_1, y) \mid a^i_1, y) = 0,
\]

or equivalently,

\[
\phi'_2(\sigma^i_2(a^i_1, y)) = \int_R \phi_2(\sigma^i_2(a^i_1, y) - \sigma^2_{2,0}(y) + \tilde{x}_1) g^i_2 f(x_1 \mid a^i_1, y) \, dx_1
\]

for every \( a^i_1 \). Since \( \inf_{a \in R^+} \phi''_2(a) > \sup_{x \in R} |\phi'_2(x)| \) by assumption, we also have

\[
\frac{\partial^2 \pi^i_2}{\partial (a^i_2)^2}(a^i_2 \mid a^i_1, y) < 0,
\]

which shows that \( \sigma^i_2(\cdot, y) \) is differentiable as a function of \( a^i_1 \) by the implicit function theorem. Likewise, agent 2’s stage 2 action satisfies

\[
\phi'_2(\sigma^i_2(a^i_1, y)) = \int_R \phi_2(-\sigma^1_{2,0}(y) + \sigma^2_{2,0}(y) - \tilde{x}_1) g^i_2 f(x_1 \mid a^i_1, y) \, dx_1
\]

for every \( a^i_1 \). Since \( \sigma^i_2(\sigma^i_1, y) = \sigma^1_{2,0}(y) \) and \( g^i_2 f(x_1 \mid \sigma^i_1, y) = g^i_1 f(x_1 \mid y) \) by definition, substitution of \( a^i_1 = \sigma^i_1 \) \( i = 1, 2 \) into (15) and (16) yields

\[
\sigma^1_{2,0}(y) = \sigma^2_{2,0}(y) = \alpha(y) \equiv (c'_2)^{-1} \left( E^{\sigma^2} \left[ \phi_2(\tilde{x}_1) \mid y \right] \right).
\]

Now let \( \pi^i_1(a^i_1) = \pi^i_1(a^i_1 \mid \sigma \) be agent i’s (overall) expected payoff when he takes \( a^i_1 \) in stage 1 and \( \sigma^i_2(a^i_1, y) \) in stage 2, while agent j plays according to his equilibrium strategy \( \sigma^j \). For \( i = 1 \), we have

\[
\pi^1_1(a^1_1)
= -c_1(a^1_1)
+ \int_R \left\{ \Phi_2(\sigma^2_2(a^1_1, f(x_1)) - \sigma^2_{2,0}(f(x_1)) + x_1) - c_2(\sigma^2_2(a^1_1, f(x_1))) \right\}
\times \phi_1(x_1 - a^1_1 + \sigma^2_1) \, dx_1.
\]
Given that $\sigma_2^1$ is differentiable in $a_1^1$ as noted above, we use the envelope theorem to differentiate $\pi_1^1$:

$$
(\pi_1^1)'(a_1^1)
= -\int R \Phi_2(\sigma_1^2(a_1^1, f(x_1)) - \sigma_2^1 f(x_1)) + x_1) \phi_1'(x_1 - a_1^1 + \sigma_1^1) dx_1
+ \int R c_2(\sigma_2^1(a_1^1, f(x_1))) \phi_1'(x_1 - a_1^1 + \sigma_1^1) dx_1 - c'_1(a_1^1).
$$

(19)

If the equilibrium stage 1 action $a_1^1 = \sigma_1^1$ is strictly positive, the FOC $(\pi_1^1)'(a_1^1) = 0$ must hold. Since $\sigma_2^0(y) = \sigma_2^2(y)$ for any $y \in Y$ by (17), this FOC is equivalent to

$$
c'_1(\sigma_1^1) = -\int R \Phi_2(x_1) \phi_1'(x_1 - \sigma_1^1 + \sigma_1^1) dx_1
+ \int R c_2(\alpha_2(\sigma_1^1, f(x_1))) \phi_1'(x_1 - \sigma_1^1 + \sigma_1^1) dx_1.
$$

Changing variables of the first integral, and then integrating it by parts, we see that this is equivalent to the first line of (2). The symmetric argument shows that the second line of (2) is equivalent to the FOC for agent 2.

**Proof of Theorem 2** Define

$$
\kappa = \min\{1, \inf_{a_1^1} c''_1(a), \inf_{a_1^1} c''_2(a), \lim_{a_1^1 \to \infty} c'_2(a)\}.
$$

Then (4) is equivalent to $\epsilon = \kappa/2$. Suppose that $\sigma_1 = (\sigma_1^1, \sigma_1^2)$ solves (2) and consider the equations

$$
\varphi_1^2(a_2^1 | a_1^1, y) \equiv \int R \phi_2(a_2^1 - \alpha_2(\sigma_1^1, y) + x_1) \phi_1^2 f(x_1 | a_1^1, y) dx_1 - c'_2(a_2^1) = 0,
$$

and

$$
\varphi_2^2(a_2^2 | a_1^1, y) \equiv \int R \phi_2(\phi_2(\sigma_1^1, y) - a_2^1 + x_1) \phi_1^2 f(x_1 | a_1^1, y) dx_1 - c'_2(a_2^2) = 0,
$$

for each $a_1^1, a_2^1 \in R$, and $y \in Y$. We have $\varphi_2^1(0 | a_1^1, y) > 0$ by $c''_2(0) = 0$ and $\phi_2 > 0$, and $\varphi_2^1(a_1^1 | a_1^1, y) < 0$ for $a_1^1$ large enough since $\lim_{a_1^1 \to \infty} c'_2(a) > \epsilon > \sup_{x \in R} \phi_2(x)$.

Furthermore, it follows from $\inf_{a_1^1} c''_2(a) > \epsilon > \sup_{x \in R} |\phi_2(x)|$ that

$$
\frac{\partial \varphi_1^2}{\partial a_2^1}(a_2^1 | a_1^1, y) = -c''_2(a_2^1) + \int R \phi_2(a_2^1 - \sigma_2^1, f(x_1) + x_1) \phi_1^2 f(x_1 | a_1^1, y) dx_1 < 0.
$$

31
Hence, there exists a unique solution to $\varphi_2^1(a_2^1 \mid a_1^1, y) = 0$, and we define $\sigma_2^1(a_1^1, y) > 0$ to be this solution. In the same manner, $\sigma_2^2(a_1^2, y)$ is defined to be the unique solution to $\varphi_2^2(a_2^2 \mid a_1^2, y) = 0$. Note now that when $a_1^1 = \sigma_1^1$ and $a_1^2 = \sigma_2^2$, $\sigma_2^2(\sigma_1^1, y) = \sigma_2^2(\sigma_1^2, y)$ solves the two equations. We can hence replace $\sigma_2(\sigma_1, y)$ in the definition of $\varphi_2(\sigma_1^1, y)$ by $\varphi_2^2(a_2^1 \mid a_1^1, y)$ and see that $\varphi_2^1(a_2^1 \mid a_1^1, y) = 0$ is equivalent to the FOC $\frac{\partial \pi_2^1}{\partial a_2^1}(a_2^1 \mid a_1^1, y) = 0$ ((15) in the proof of Theorem 1) of agent 1’s stage 2 payoff maximization problem. Likewise, $\varphi_2^2(a_2^2 \mid a_1^2, y) = 0$ is equivalent to the FOC for agent 2’s stage 2 maximization problem. To see that $a_2^1 = \sigma_2^1(a_1^1, y)$ does maximize 1’s payoff, it suffices to note that

$$
\frac{\partial^2 \pi_2^1}{\partial (a_2^1)^2}(a_2^1 \mid a_1^1, y) = -c'_{2}(a_2^1) + \int_R \phi_{2}^1(a_2^1 - \sigma_2^1(y) + x_1) g_1 \phi'(x_1 \mid a_1^1, y) dx_1
$$

$$
< -\kappa + \epsilon < 0.
$$

The same observation holds for agent 2.

We now turn to the analysis of stage 1 effort. As in the proof of Theorem 1, denote by $\pi_1^i(a_1^i)$ agent $i$’s overall payoff when he takes action $a_1^i$ in stage 1 and chooses $\sigma_2^j(a_1^1, y)$ in stage 2, and agent $j$ takes action $\sigma_1^j$ in stage 1 and chooses $\sigma_2^j(\sigma_1^j, y)$ in stage 2. Define

$$
\varphi_1^i(a_1^i) = -c'_1(a_1^i) + (\phi_1 \ast \phi_2)(a_1^i - \sigma_1^i)
$$

$$
+ \int_R c_2(\alpha_2(a_1^i, \sigma_1^i, f(x_1))) \phi'_1(x_1 - a_1^i + \sigma_1^i) dx_1,
$$

$$
\varphi_2^i(a_2^i) = -c'_2(a_1^i) + (\phi_1 \ast \phi_2)(\sigma_1^i - a_1^i)
$$

$$
- \int_R c_2(\alpha_2(\sigma_1^i, a_1^i, f(x_1))) \phi'_1(x_1 - \sigma_1^i + a_1^i) dx_1.
$$

By assumption, $a_1^i = \sigma_1^i$ solves $\varphi_1^i(a_1^i) = 0$. Furthermore, $\varphi_1^i(a_1^i) = (\pi_1^i)'(a_1^i)$ as seen in the proof of Theorem 1 so that $\sigma_1^i$ is a solution to the FOC of agent $i$’s payoff maximization problem. In what follows, We will show $(\pi_1^i)' = (\pi_1^i)'' < 0$ and hence $\sigma_1^i$ is indeed the maximizer of $\pi_1^i$.

Since $\sigma_1^i$ is differentiable with respect to $a_1^i$ as noted in the proof of Theorem 1,
we can differentiate (19) to obtain

\[(\varphi_1')(a_1^1) = -c''(a_1^1) \]

\[- \int_R \left\{ \phi_2 (\sigma_1^2(a_1^1, f(x_1)) - \sigma_2^2(f(x_1)) + x_1) - c_2' (\sigma_2^2(a_1^1, f(x_1))) \right\} \]

\[\times \partial g_{a_1^1}(a_1^1, f(x_1)) \phi_1'(x_1 - a_1^1 + \sigma_1^2) dx_1 \]

\[+ \int_R \left\{ \Phi_2 (\sigma_1^2(a_1^1, f(x_1)) - \sigma_2^2(f(x_1)) + x_1) - c_2(\sigma_2^2(a_1^1, f(x_1))) \right\} \]

\[\times \phi_1''(x_1 - a_1^1 + \sigma_1^2) dx_1. \]

Note now that for any \( y \in Y \), we have \( c_2'(\sigma_2^2(a_1^1, y)) \leq \epsilon \) by (16) and \( c_2(\sigma_2^2(a_1^1, y)) \leq 1 \) by the above observation that \( \sigma_2^2(a_1^1, y) \) maximizes \( \pi_2(\cdot \mid a_1^1, y) \). Hence,

\[|\phi_2(\sigma_2^2(a_1^1, f(x_1)) - \sigma_2^2(f(x_1)) + x_1) - c_2'(\sigma_2^2(a_1^1, f(x_1)))| \leq \epsilon, \]

and

\[|\Phi_2(\sigma_2^2(a_1^1, f(x_1)) - \sigma_2^2(f(x_1)) + x_1) - c_2(\sigma_2^2(a_1^1, f(x_1)))| \leq 1. \]

It follows that

\[(\varphi_1')(a_1^1) \leq -c''(a_1^1) + \epsilon \int_R \left| \partial g_{a_1^1}(a_1^1, f(x_1)) \right| \left| \phi_1'(x_1 - a_1^1 + \sigma_1^2) \right| dx_1 \]

\[+ \int_R \left| \phi_1''(x_1 - a_1^1 + \sigma_1^2) \right| dx_1. \]

(20)

Now take \( y \in Y \) such that \( f^{-1}(y) \) has positive measure. Then for \( x_1 \in f^{-1}(y) \),

\[
\frac{\partial g_{a_1}^{\sigma,f}(x_1 \mid a_1^1, y)}{\partial a_1^1} = \frac{-\phi_1'(x_1 - a_1^1 + \sigma_1^2)}{\int_{f^{-1}(y)} \phi_1(\hat{x}_1 - a_1^1 + \sigma_1^2) d\hat{x}_1} \]

\[+ \frac{\phi_1'(x_1 - a_1^1 + \sigma_1^2) \int_{f^{-1}(y)} \phi_1'(\hat{x}_1 - a_1^1 + \sigma_1^2) d\hat{x}_1}{\left\{ \int_{f^{-1}(y)} \phi_1(\hat{x}_1 - a_1^1 + \sigma_1^2) d\hat{x}_1 \right\}^2}, \]

and hence

\[
\int_R \left| \frac{\partial g_{a_1}^{\sigma,f}(x_1 \mid a_1^1, y)}{\partial a_1^1} \right| dx_1 \leq 2 \frac{\int_{f^{-1}(y)} \phi_1'(x_1 - a_1^1 + \sigma_1^2) dx_1}{\int_{f^{-1}(y)} \phi_1(x_1 - a_1^1 + \sigma_1^2) dx_1}. \]

Now consider the distribution of \( x_1 \) under the stage 1 effort profile \((a_1^1, \sigma_1^2)\) and denote the corresponding expectation by \( \hat{E} \). If we let

\[q(x_1) = \frac{|\phi_1'(x_1 - a_1^1 + \sigma_1^2)|}{\phi_1(x_1 - a_1^1 + \sigma_1^2)}, \]

33
then the above inequality can be written as
\[ \int_{\mathbb{R}} \left| \frac{\partial \varphi^1}{\partial a_1^1} (x_1 | a_1^1, y) \right| dx_1 \leq \hat{E}[q(\bar{x}_1) | y]. \]

On the other hand,
\[ \frac{\partial \varphi^1}{\partial a_1^1} (a_2^1 | a_1^1, y) = \int_{\mathbb{R}} \phi_2 (a_2^1 - \sigma_{2,0}^2 (y) + x_1) \frac{\partial \varphi^1}{\partial a_1^1} (x_1 | a_1^1, y) dx_1, \]
so that
\[ \left| \frac{\partial \varphi^1}{\partial a_1^1} (a_2^1 | a_1^1, y) \right| < \epsilon \int_{\mathbb{R}} \left| \frac{\partial \varphi^1}{\partial a_1^1} (x_1 | a_1^1, y) \right| dx_1 \leq 2 \epsilon \hat{E}[q(\bar{x}_1) | y]. \]

Therefore,
\[ \frac{1}{2 \epsilon} \int_{\mathbb{R}} \left| \frac{\partial \varphi^1}{\partial a_1^1} (a_2^1 | a_1^1, y = f(x_1)) \right| \left| \phi_1'(x_1 - a_1^1 + \sigma_1^2) \right| dx_1 \]
\[ \leq \int_{\mathbb{R}} \hat{E}[q(\bar{x}_1) | \bar{y} = f(x_1)] q(x_1) \phi_1 (x_1 - a_1^1 + \sigma_1^2) dx_1 \]
\[ = \hat{E} \left[ \hat{E}[q(\bar{x}_1) | \bar{y}] q(\bar{x}_1) \right] \]
\[ \leq \hat{E} \left[ \hat{E}[q(\bar{x}_1) | \bar{y}]^2 \right]^{1/2} \hat{E} [q(\bar{x}_1)^2]^{1/2} \]
\[ \leq \hat{E} \left[ \hat{E}[q(\bar{x}_1)^2 | \bar{y}] \right]^{1/2} \hat{E} [q(\bar{x}_1)^2]^{1/2} \]
\[ = \hat{E} [q(\bar{x}_1)^2] = \int_{\mathbb{R}} \left| \phi_1'(x_1) \right|^2 \phi_1(x_1) dx_1 < \epsilon, \]
where the fourth line follows from Schwartz’ inequality and the fifth line from Jensen’s inequality. Using the implicit function theorem, we see that the second term on the right-hand side of (20) can be evaluated as:
\[ \epsilon \int_{\mathbb{R}} \left| \frac{\partial \varphi^1}{\partial a_1^1} (a_1^1, f(x_1)) \right| \left| \phi_1'(x_1 - a_1^1 + \sigma_1^2) \right| dx_1 \]
\[ = \epsilon \int_{\mathbb{R}} \left| \frac{\partial \varphi^1}{\partial a_1^1} (\sigma_1^2 (a_1^1, f(x_1)) | a_1^1, y = f(x_1)) \right| \left| \phi_1'(x_1 - a_1^1 + \sigma_1^2) \right| dx_1 \]
\[ \leq \frac{\epsilon}{\kappa - \epsilon} \int_{\mathbb{R}} \left| \frac{\partial \varphi^1}{\partial a_1^1} (\sigma_1^2 (a_1^1, f(x_1)) | a_1^1, f(x_1)) \right| \left| \phi_1'(x_1 - a_1^1 + \sigma_1^2) \right| dx_1 \]
\[ \leq \frac{2 \epsilon^3}{\kappa - \epsilon} \]
Hence,
\[ \left( \varphi^1 \right)'(a_1^1) \leq -\kappa + \frac{2 \epsilon^3}{\kappa - \epsilon} + \epsilon < 0. \]

This proves the claim. //
Proof of Theorem 3  Suppose that $\sigma_1^1 = \sigma_2^2$. We first show that $\alpha_2(\sigma_1, f(x_1)) = \alpha_2(\sigma_1, f(-x_1))$ for any $x_1$. This would hold trivially if $f$ is even since then $f(x_1) = f(-x_1)$. If $f$ is odd, then $g_1^{\sigma,f}(x_1 \mid y) = g_1^{\sigma,f}(-x_1 \mid -y)$, and hence the symmetry of $\phi_2$ implies that

$$
\alpha_2(\sigma_1, y) = (c'_2)^{-1}\left(\int_R \phi_2(x_1) g_1^{\sigma,f}(x_1 \mid y) \, dx_1\right)
= (c'_2)^{-1}\left(\int_R \phi_2(-x_1) g_1^{\sigma,f}(-x_1 \mid -y) \, dx_1\right)
= \alpha_2(\sigma_1, -y).
$$

It follows that $\alpha_2(\sigma_1, f(-x_1)) = \alpha_2(\sigma_1, -f(x_1)) = \alpha_2(\sigma_1, f(x_1))$. With this equality, $\sigma_1^1 = \sigma_2^2 = \alpha_1^*$ solves (2) since

$$
\int_R c_2(\alpha_2(\sigma_1, f(x_1))) \phi'_1(x_1) \, dx_1 
= \int_0^\infty c_2(\alpha_2(\sigma_1, f(x_1))) \phi'_1(x_1) \, dx_1 + \int_{-\infty}^0 c_2(\alpha_2(\sigma_1, f(x_1))) \phi'_1(x_1) \, dx_1 
= \int_0^\infty c_2(\alpha_2(\sigma_1, f(x_1))) \phi'_1(x_1) \, dx_1 - \int_0^\infty c_2(\alpha_2(\sigma_1, f(-x_1))) \phi'_1(x_1) \, dx_1 
= 0.
$$

This completes the proof. //

Lemma 17 Suppose that Assumptions 1, 3 and 4 hold and that $\lim_{a \to -\infty} c_i'(a) > 2(\phi_1 * \phi_2)(0)$ for $i = 1, 2$. Then for any $\sigma_1$ that solves (2) and any $a_2$ such that $a_2^1 = a_2^2$, the principal’s payoff function satisfies

$$
V((\alpha_1^*, \alpha_1^*), a_2) \geq V(\sigma_1, a_2).
$$

Proof of Lemma 17 Fix $a_2 \in \mathbb{R}_+^2$ such that $a_2^1 = a_2^2$. Since $h$ is continuous, the inverse image $h^{-1}\{0\}$ is closed. Furthermore, it is non-empty since $(a_*^1, a_*^1) \in h^{-1}\{0\})$. To see that it is bounded, note that

$$
h(a_1) \geq c'_1(a_1^1) + c'_1(a_1^2) - 2(\phi_1 * \phi_2)(0)
$$

by Assumption 3, and that $(c'_1)^{-1}(2(\phi_1 * \phi_2)(0)) < \infty$ by assumption. Hence, $h(a_1) > 0$ for any $a_1$ such that $\max\{a_1^1, a_1^2\} > (c'_1)^{-1}(2(\phi_1 * \phi_2)(0))$ by Assumption
1. It follows that the continuous function \( V(\cdot, a_2) \) on the compact set \( h^{-1}(\{0\}) = \{ a_1 \in \mathbb{R}^2_+ : h(a_1) = 0 \} \) achieves a maximum. Let \( \bar{a}_1 = (\bar{a}_1^1, \bar{a}_1^2) \in h^{-1}(\{0\}) \) be any maximizer of \( V(\cdot, a_2) \) in \( h^{-1}(\{0\}) \). We show that \( \bar{a}_1 = (a_1^*, a_1^*) \). Suppose that \( \bar{a}_1^1 < \bar{a}_1^2 \). Since \( \frac{\partial h}{\partial a_1^1} \neq 0 \) by (6), the implicit function theorem shows that there exists a function \( \gamma \) defined in a neighborhood of \( \bar{a}_1^1 \) such that \( h(a_1^1, \gamma(a_1^1)) = 0 \). Furthermore, \( \gamma \) is differentiable at \( \bar{a}_1^1 \) and the derivative \( \gamma'(\bar{a}_1^1) \) is given by the left-hand side of (6) with \( \bar{a}_1^1 \) replacing \( a_1^1 \). Now let \( \delta(a_1^1) = V((a_1^1, \gamma(a_1^1)), a_2) \). \( \delta \) is also differentiable at \( \bar{a}_1^1 \) and its derivative is given by

\[
\delta'(\bar{a}_1^1) = \frac{\partial V}{\partial a_1^1}(\bar{a}_1, a_2) + \frac{\partial V}{\partial a_1^2}(\bar{a}_1, a_2) \gamma'(\bar{a}_1^1).
\]

It can be readily verified that Assumption 4 implies \( \delta'(\bar{a}_1^1) > 0 \). This contradicts our assumption that \( V \) is maximized at \( \bar{a}_1 \) in \( h^{-1}(\{0\}) = 0 \). The symmetric argument shows that it cannot be maximized at \( \bar{a} \) such that \( \bar{a}_1^1 > \bar{a}_1^2 \) either. Hence, we must have \( \bar{a}_1^1 = \bar{a}_1^2 = a_1^* \). //

**Proof of Theorem 8** Let \( f \) be any feedback policy that admits a PBE \( \sigma \) for which (2) holds. As in the proof of Theorem 5, Jensen’s inequality and the law of iterated expectation applied to (5) imply that the expected stage 2 effort satisfies

\[
E^\sigma f[a_2(\sigma_1, \tilde{y})] = E^\sigma f \left[ (c_2')^{-1} \left( E^\sigma f[\phi_2(\tilde{x}_1) | \tilde{y}] \right) \right] \\
\leq (c_2')^{-1} \left( E^\sigma f \left[ E^\sigma f[\phi_2(\tilde{x}_1) | \tilde{y}] \right] \right) \\
= (c_2')^{-1} \left( E^\sigma f[\phi_2(\tilde{x}_1)] \right) \\
= (c_2')^{-1} \left( (\phi_1 \ast \phi_2)(\sigma_1^1 - \sigma_1^2) \right) \\
\leq (c_2')^{-1} \left( (\phi_1 \ast \phi_2)(0) \right) = a_2^*,
\]

where the last inequality follows from 3. It hence follows from Assumption 2 that

\[
v(\sigma, f) = E^\sigma f \left[ V \left( \sigma_1, a_2^1 = a_2^2 = \alpha_2(\sigma_1, \tilde{y}) \right) \right] \leq V(\sigma_1, (a_2^*, a_2^*)).
\]

Since \( \sigma_1 \) solves (2) by assumption,

\[
V(\sigma_1, (a_2^*, a_2^*)) \leq V((a_1^*, a_1^*), (a_2^*, a_2^*))
\]

by Lemma 17. Since the right-hand side of the above inequality equals the principal’s expected payoff in the symmetric PBE under the no-feedback policy, the desired conclusion follows. //
Proof of Theorem 9  We first show that Assumption 5 implies

\[ P(|\tilde{\zeta}_2| \geq \kappa) = \min_{\delta \in \mathbb{R}} P(|\tilde{\zeta}_2 + \delta| \geq \kappa) \text{ for any } \kappa > 0. \]  

Let \( \delta > 0 \) and \( \kappa > 0 \) be given. When \( \delta < 2\kappa \), we have

\[
P(|\tilde{\zeta}_2| < \kappa) - P(|\tilde{\zeta}_2 + \delta| < \kappa) = - \int_{-\kappa-\delta}^{-\kappa} \phi_2(x) \, dx + \int_{-\kappa}^{\kappa-\delta} \phi_2(x) \, dx \]

\[ > - \delta \phi_2(-\kappa) + \delta \phi_2(\kappa) \]

\[ = 0. \]

On the other hand, when \( \delta > 2\kappa \), we have

\[
P(|\tilde{\zeta}_2| < \kappa) - P(|\tilde{\zeta}_2 + \delta| < \kappa) = \int_{-\kappa}^{\kappa-\delta} \phi_2(x) \, dx - \int_{-\kappa-\delta}^{-\kappa} \phi_2(x) \, dx \]

\[ > 2\kappa \phi_2(\kappa) - 2\kappa \phi_2(\kappa - \delta) \]

\[ > 0. \]

The similar argument proves (21) when \( \delta < 0 \).

We now show that the expected stage 2 effort implied by \( \sigma \) is less than or equal to that implied by the symmetric PBE under the full-feedback policy:

\[ E^{\sigma,f}[\alpha_2(\sigma_1, y)] \leq a^{**}_2 \equiv \int_{\mathbb{R}} (c_2')^{-1}(\phi_2(x_1)) \phi(x_1) \, dx_1. \]  

By the same logic as in the proof of Theorem 8, it would then follow from Lemma 7 and Assumption 2 that \( v(\sigma, f) \) is \( \leq \) the principal’s expected payoff in the symmetric PBE under the full-feedback policy.

Note that since \( E^{\sigma,f}[\alpha_2(\sigma_1, y)] \leq E^{\sigma,f}[(c_2')^{-1}(\phi_2(\tilde{x}_1))] \) as in the proof of Theorem 6, (22) is implied by

\[ E^{\sigma,f}[(c_2')^{-1}(\phi_2(\tilde{x}_1))] \leq a^{**}_2. \]  

Let \( \eta_2 : [0, \phi_2(0)] \to \mathbb{R}_+ \) be the inverse of the restriction of \( \phi_2 \) to \( \mathbb{R}_+ \). In other words, for each \( u \in [0, \phi_2(0)] \), \( \eta_2(u) \geq 0 \) is the unique number such that \( \phi_2(\eta_2(u)) = u \).

Note that \( \eta_2 \) is well-defined under Assumption 5. Given any \( \delta \in \mathbb{R} \), let the function \( G(\cdot \mid \delta) : [0, \phi_2(0)] \to \mathbb{R}_+ \) be defined by \( G(u \mid \delta) = 1 - \Phi_2(\eta_2(u) - \delta) + \Phi_2(-\eta_2(u) - \delta) = P(|\zeta_2 + \delta| \geq \eta_2(u)) \). It is easy to verify that \( G(\cdot \mid \delta) \) is a distribution function.
over \([0, \phi_2(0)]\) since it is increasing, and satisfies \(G(0 \mid \delta) = 0\) and \(G(\phi_2(0) \mid \delta) = 1\).

If we write \(\delta = a_1^* - a_1^0\) and \(u = \phi_2(x_1)\), then

\[
E[(c_2')^{-1}(\phi_2(\phi_2(x_1))) \mid a_1] = \int_{\mathbb{R}} (c_2')^{-1}(\phi_2(x_1)) \phi_1(x_1 - \delta) \, dx_1
\]

\[
= \int_0^{\phi_2(0)} (c_2')^{-1}(\phi_2(x_1)) \phi_1(\eta_2(u) - \delta) (-\eta_2'(u)) \, du
\]

\[
+ \int_0^{\phi_2(0)} (c_2')^{-1}(\phi_2(x_1)) \phi_1(-\eta_2(u) - \delta) (-\eta_2'(u)) \, du
\]

\[
= \int_0^{\phi_2(0)} (c_2')^{-1}(\phi_2(x_1)) \, dG(u \mid \delta).
\]

By (21), \(G(u \mid \delta) = P(|\zeta_2 + \delta| \geq \eta_2(u)) \geq P(|\zeta_2| \geq \eta_2(u)) = G(u \mid 0)\) for any \(u \in [0, \phi_2(0)]\) and \(\delta \in \mathbb{R}\) so that \(G(u \mid 0)\) first-order stochastically dominates \(G(u \mid \delta)\) with \(\delta \neq 0\). Since \((c_2')^{-1}\) is increasing, it follows that

\[
\int_0^{\phi_2(0)} (c_2')^{-1}(\phi_2(x_1)) \, dG(u \mid \delta) \leq \int_0^{\phi_2(0)} (c_2')^{-1}(\phi_2(x_1)) \, dG(u \mid 0).
\]

Changing variables back to \(x_1\), we see that the right-hand side of this inequality equals \(a_2^*\).

---

**Proof of Theorem 12**  For any \(\hat{r} \in (r, (c_2')^{-1}(\phi_2(0)))\), define

\[
p = \int_{\{x_1: \phi_2(x_1) \geq c_2'(\hat{r})\}} \phi_1(x_1) \, dx_1,
\]

\[
x = \frac{1}{1 - p} \int_{\{x_1: \phi_2(x_1) < c_2'(\hat{r})\}} \phi_2(x_1) \phi_1(x_1) \, dx_1,
\]

\[
y = \frac{1}{p} \int_{\{x_1: \phi_2(x_1) \geq c_2'(\hat{r})\}} \phi_2(x_1) \phi_1(x_1) \, dx_1.
\]

Now consider the even feedback policy \(f\) that only reveals whether \(\phi_2(x_1) \geq c_2'(\hat{r})\) or not. We can express the expected policy effort in the symmetric PBE \(\sigma\) under \(f\) as

\[
E^{\sigma,f}[\left(c_2'\right)^{-1}(E^{\sigma,f}[\phi_2(\tilde{x}_1) \mid \tilde{y}])]
\]

\[
= E^{\sigma,f}[\left(c_2'\right)^{-1}(E^{\sigma,f}[\phi_2(\tilde{x}_1) \mid \tilde{y}]) \mid \phi_2(\tilde{x}_1) \geq c_2'(\hat{r})] P^{\sigma,f}(\phi_2(\tilde{x}_1) \geq c_2'(\hat{r}))
\]

\[
+ E^{\sigma,f}[\left(c_2'\right)^{-1}(E^{\sigma,f}[\phi_2(\tilde{x}_1) \mid \tilde{y}]) \mid \phi_2(\tilde{x}_1) < c_2'(\hat{r})] P^{\sigma,f}(\phi_2(\tilde{x}_1) < c_2'(\hat{r}))
\]

\[
= (1 - p)(c_2')^{-1}(x) + p(c_2')^{-1}(y),
\]

38
where the second equality follows since the functions inside the expectations are constant over the conditioning events by assumption. On the other hand, the stage 2 effort in the symmetric PBE under the no-feedback policy can be expressed as

\[(c'_2)^{-1}((1 - p) x + py).\]

Since \( x \rightarrow (\phi_1 \ast \phi_2)(0) > c'_2(r) \) as \( r \rightarrow (c'_2)^{-1}(\phi_2(0)) \), we can take \( r \) close enough to \((c'_2)^{-1}(\phi_2(0))\) so that \( x > c'_2(r) \). Since then \( x, y \in [c'_2(r), \phi_2(0)] \) over which \((c'_2)^{-1}\) is convex by assumption, we have

\[(1 - p) (c'_2)^{-1}(x) + p(c'_2)^{-1}(y) \geq (c'_2)^{-1}((1 - p) x + py).\]

This proves the claim. //

**Proof of Theorem 13** The proof consists of three steps. In the first step, we specify the effort level in every stage contingent on each possible history. The second step shows that if there exists a symmetric pure PBE, then the agent’s equilibrium effort level must be as specified in Step 1. In the third step, it is shown that this is indeed an equilibrium effort level. Define

\[\kappa = \min \left\{ 1, \frac{1}{T} \min_{1 \leq t \leq T} \inf_{a_t \in \mathbb{R}} c'_t(a_t) \right\} > 0,\]

and take \( \epsilon > 0 \) such that

\[
\max \left\{ c_t \left( (c'_t)^{-1}(\epsilon T) \right), c_t \left( (c'_t)^{-1}(-\epsilon T) \right) \right\} < 1 \quad \text{for } t = 1, \ldots, T,
\]

(25)

\[
\epsilon < 2^{-T} \kappa, \quad \text{and}
\]

\[
\epsilon < \frac{1}{T} \min_{1 \leq t \leq T} \min \left\{ \lim_{a \to -\infty} c'_t(a), \lim_{a \to -\infty} |c'_t(-a)| \right\}.
\]

**Step 1.** For each \( z_{t-1} \in Z_{t-1}(f) \) and \( t = 1, \ldots, T \), define

\[\alpha_t(z_{t-1}) = (c'_t)^{-1} \left( \int_{\mathbb{R}^{t-1}} \phi_T(\Delta T_{t-1}) \prod_{s=t}^{T-1} \phi_s(x_s) g^a_{t-1}(\omega_{t-1} \mid z_{t-1}) d\omega_{T-1} \right).\]

Let \( g^a_{t-1}(\omega_{t-1} \mid b^1_{t-1}, z_{t-1}) \) denote the density of \( \omega_{t-1} \) conditional on the sequences \( z_{t-1} = (y_1, \ldots, y_{t-1}) \) and \( b^1_{t-1} = (a^1_1, \ldots, a^1_{t-1}) \) of announcements and actions, respectively, provided that agent 2 chooses \( \alpha_s(z_{s-1}) \) in stage \( s = 1, \ldots, t - 1 \):

\[g^a_{t-1}(\omega_{t-1} \mid b^1_{t-1}, z_{t-1}) = \frac{\prod_{s=1}^{t-1} \phi_s(x_s - a^1_s + \alpha_s(z_{s-1}))}{\int_{X^1_{t-1}(z_{t-1})} \prod_{s=1}^{t-1} \phi_s(x'_s - a^1_s + \alpha_s(z_{s-1})) d\omega'_{t-1}}.\]
if \( \omega_{t-1} \in X^f_{t-1}(z_{t-1}) \) and \( g^\alpha_{t-1}(\omega_{t-1} \mid b^1_{t-1}, z_{t-1}) = 0 \) otherwise. Note that for any \( \omega_{t-1} \in X^f_{t-1}(z_{t-1}) \) and \( u = 1, \ldots, t-1, \)

\[
\frac{\partial g^\alpha_{t-1}(\omega_{t-1} \mid b^1_{t-1}, z_{t-1})}{\partial a^1_u} = -\frac{\phi'_u(x_u - a^1_u + \alpha_u(z_{u-1}))}{\phi_u(x_u - a^1_u + \alpha_u(z_{u-1}))} \prod_{s=1}^{t-1} \phi_s(x_s - a^1_s + \alpha_s(z_{s-1})) \int_{X^f_{t-1}(z_{t-1})} \prod_{s=1}^{t-1} \phi_s(x'_s - a^1_s + \alpha_s(z_{s-1})) \, d\omega'_t
\]

\[
+ \frac{\prod_{s=1}^{t-1} \phi_s(x_s - a^1_s + \alpha_s(z_{s-1}))}{(\int_{X^f_{t-1}(z_{t-1})} \prod_{s=1}^{t-1} \phi_s(x'_s - a^1_s + \alpha_s(z_{s-1})) \, d\omega'_t)^2} \times \int_{X^f_{t-1}(z_{t-1})} \frac{\phi'_u(x'_u - a^1_u + \alpha_u(z_{u-1}))}{\phi_u(x'_u - a^1_u + \alpha_u(z_{u-1}))} \prod_{s=1}^{t-1} \phi_s(x'_s - a^1_s + \alpha_s(z_{s-1})) \, d\omega'_t.
\]

Hence, when (10) holds,

\[
\int_{X^f_{t-1}(z_{t-1})} \left| \frac{\partial g^\alpha_{t-1}(\omega_{t-1} \mid b^1_{t-1}, z_{t-1})}{\partial a^1_u} \right| \, d\omega_{t-1} \leq 2 \int_{X^f_{t-1}(z_{t-1})} \frac{\phi'_u(x'_u - a^1_u + \alpha_u(z_{u-1}))}{\phi_u(x'_u - a^1_u + \alpha_u(z_{u-1}))} \prod_{s=1}^{t-1} \phi_s(x_s - a^1_s + \alpha_s(z_{s-1})) \, d\omega_{t-1}
\]

\[
\leq 2\epsilon.
\]

Now for each \( b^1_{T-1} = (a^1_1, \ldots, a^1_{T-1}) \), \( z_{T-1} \in Z_{T-1}(f) \), and \( a^1_T \in R \), let

\[
\varphi^1_T(a^1_T \mid b^1_{T-1}, z_{T-1}) = -c_T(a^1_T) + \int_{R^{T-1}} \phi_T \left( a^1_T - \alpha_T(z_{T-1}) + \Delta_{T-1} \right) \times g^\alpha_{T-1}(\omega_{T-1} \mid b^1_{T-1}, z_{T-1}) \, d\omega_{T-1}.
\]

Note that \( \varphi^1_T \) is continuous in \( a^1_T \), and that \( \varphi^1_T(-a^1_T \mid b^1_{T-1}, z_{T-1}) > 0 \), and \( \varphi^1_T(a^1_T \mid b^1_{T-1}, z_{T-1}) < 0 \) for \( a^1_T \) large enough by (25). Furthermore,

\[
\frac{\partial \varphi^1_T}{\partial a^1_T}(a^1_T \mid b^1_{T-1}, z_{T-1})
\]

\[
= -c''(a^1_T) + \int_{R^{T-1}} \phi'_T \left( a^1_T - \alpha_T(z_{T-1}) + \Delta_{T-1} \right) \times g^\alpha_{T-1}(\omega_{T-1} \mid b^1_{T-1}, z_{T-1}) \, d\omega_{T-1}
\]

\[
\leq -\kappa + \epsilon < 0
\]

40
for any $a^1_T \in R$. Hence, there exists a unique $a^1_T \in R$ that solves $\varphi^1_T(a^1_T | b^1_{T-1}, z_{T-1}) = 0$. We define $\sigma^1_T(b^1_{T-1}, z_{T-1})$ to be this solution. Agent 2’s contingent action $\sigma^2_T$ in stage $T$ is defined in a similar manner: For each $b^2_{T-1} = (a^2_1, \ldots, a^2_{T-1})$, $z_{T-1} \in Z_{T-1}(f)$, and $a^2_T \in R$, let $\sigma^2_T(b^2_{T-1}, z_{T-1})$ be the unique solution to $\varphi^2_T(a^2_T | b^2_{T-1}, z_{T-1}) = 0$, where

$$\varphi^2_T(a^2_T | b^2_{T-1}, z_{T-1}) = -c^2_T(a^2_T) + \int_{R^{T-1}} \phi_T(-\alpha_T(z_{T-1}) + a^2_T - \Delta_{T-1}) \times g^2_{T-1}(\omega_{T-1} | b^2_{T-1}, z_{T-1}) d\omega_{T-1}.$$

We now show that $\sigma^i_T$ defined above satisfies (i) and (ii) below.

(i) $c_T(\sigma^i_T(b^i_{T-1}, z_{T-1})) < 1$ and $|c'_T(\sigma^i_T(b^i_{T-1}, z_{T-1}))| < 1$ for any $(b^i_{T-1}, z_{T-1})$.

(ii) For $u = 1, \ldots, T - 1$, $\sigma^i_T$ is differentiable as a function of $a^i_u$, and

$$\left| \frac{\partial \sigma^i_T}{\partial a^i_u}(b^i_{T-1}, z_{T-1}) \right| < 1 \quad \text{for any } (b^i_{T-1}, z_{T-1}).$$

It is clear from the definition of $\sigma^i_T$ and (10) that $|c'_T(\sigma^i_T(h^i_{T-1}))| \leq \epsilon < 1$ and hence by (25) that

$$c_T(\sigma^i_T(h^i_{T-1})) \leq \max \{ c_T((c'_T)^{-1}(1)) , c_T((c'_T)^{-1}(-1)) \} < 1.$$

Since $\frac{\partial \varphi^1_T}{\partial a^1_T}(a^1_T | b^1_{T-1}, z_{T-1}) > 0$ as noted above, $\sigma^1_T$ is differentiable with respect to $a^1_u$ by the implicit function theorem, and the derivative is given by

$$\frac{\partial \sigma^1_T}{\partial a^1_u}(b^1_{T-1}, z_{T-1}) = -\frac{\frac{\partial \varphi^1_T}{\partial a^1_u}(\sigma^1_T(b^1_{T-1}, z_{T-1}) | b^1_{T-1}, z_{T-1})}{\frac{\partial \varphi^1_T}{\partial a^1_u}(\sigma^1_T(b^1_{T-1}, z_{T-1}) | b^1_{T-1}, z_{T-1})}.$$ 

Since

$$\frac{\partial \varphi^1_T}{\partial a^1_u}(b^1_{T-1}, z_{T-1}) = \int_{R^{T-1}} \phi_T(\sigma^1_T - \alpha_T + \Delta_{T-1}) \frac{\partial g^2_{T-1}(\omega_{T-1} | b^1_{T-1}, z_{T-1}) d\omega_{T-1},$$

we have $\left| \frac{\partial \varphi^1_T}{\partial a^1_u} \right| \leq 2\epsilon^2$ by (26). Hence, it follows from (27) that

$$\left| \frac{\partial \sigma^1_T}{\partial a^1_u}(b^1_{T-1}, z_{T-1}) \right| \leq \frac{2\epsilon^2}{\kappa - \epsilon} < 1.$$

As an induction hypothesis, fix $t < T$ and suppose that we have defined effort levels $\sigma^1_T, \ldots, \sigma^i_{i+1}$ for which (i) and (ii) below hold $(s = t + 1, \ldots, T)$:

(i) $c_s(\sigma^i_s(b^i_{s-1}, z_{s-1})) < 1$ and $|c'_s(\sigma^i_s(b^i_{s-1}, z_{s-1}))| < 1$ for any $(b^i_{s-1}, z_{s-1})$. 

41
(ii) For $u = 1, \ldots, s - 1$, $\sigma^i_s(b^i_{s-1}, z_{s-1})$ is differentiable as a function of $a^1_u$, and

$$\left| \frac{\partial \sigma^i_s(b^i_{s-1}, z_{s-1})}{\partial a^1_u} \right| < 1 \quad \text{for any } (b^i_{s-1}, z_{s-1}).$$

For each $b^i_t$ and $z_t$, we first define $\sigma^i_{s,t}$ ($s > t$) as follows: For $s = t + 1$,

$$\sigma^i_{t+1,t}(b^i_t, z_t) = \sigma^i_{t+1}(b^i_t, z_t),$$

and for each $s = t + 2, \ldots, T$, define $\sigma^i_{s,t}$ recursively by

$$\sigma^i_{s,t}(b^i_t, z_{s-1}) = \sigma^i_s((b^i_t, \sigma^i_{t+1,t}(b^i_t, z_t), \ldots, \sigma^i_{s-1,t}(b^i_t, z_{s-2})), z_{s-1}).$$

The interpretation is that $\sigma^i_{s,t}(b^i_t, z_{s-1})$ is agent $i$’s action in stage $s$ induced by $\sigma^i_{t+1}, \ldots, \sigma^i_s$ after the sequence of actions $b^i_t$ in stages $1, \ldots, t$ and announcements $z_{s-1}$ after stages $1, \ldots, s - 1$. It can be verified that for any $u \leq t < s$,

$$\frac{\partial \sigma^i_{s,t}}{\partial a^1_u} = \sum_{k=0}^{s-t-1} \sum_{t_{\tau_1} < t_{\tau_2} < \ldots < t_{\tau_k} < s} \frac{\partial \sigma^i_{s,t}}{\partial a^1_{t_{\tau_k}}} \frac{\partial \sigma^i_{t_{\tau_k}}}{\partial a^1_{t_{\tau_{k-1}}}} \cdots \frac{\partial \sigma^i_{t_{\tau_2}}}{\partial a^1_{t_{\tau_1}}} \frac{\partial \sigma^i_{t_{\tau_1}}}{\partial a^1_u}.$$

We now define $\sigma^1_t$ as follows. For $b^1_{t-1} = (a^1_1, \ldots, a^1_{t-1})$, $z_{t-1} \in Z_{t-1}(f)$, and $a^1_t \in \mathbb{R}$, let

$$\varphi^1_t(a^1_t | b^1_{t-1}, z_{t-1}) = -c_t^i(a^1_t)$$

$$+ \int_{\mathbb{R}^{T-1}} \Phi_T(\sigma^1_{T,t}((b^1_{t-1}, a^1_t), z_{T-1}) - \alpha_T(z_{T-1}) + \Delta_{T-1})$$

$$- \sum_{s=t+1}^{T} c_s(\sigma^1_s((b^1_{t-1}, a^1_t), z_{s-1})) \}$$

$$\times \phi_t'(x_t - a^1_t + \alpha_t(z_{t-1}))$$

$$\times \prod_{s=t+1}^{T-1} \phi_s(x_s - \sigma^1_{s,t}((b^1_{t-1}, a^1_t), z_{s-1}) + \alpha_t(z_{s-1}))$$

$$\times g^s_{t-1}(\omega_{t-1} | b^1_{t-1}, z_{t-1}) d\omega_{T-1},$$

where $z_{s-1} = (z_{t-1}, f_t(\omega_t), \ldots, f_{s-1}(w_{s-1}))$ for $s = t + 1, \ldots, T$. By the induction
In order to evaluate $\frac{\partial \varphi_1}{\partial a_t^1}(a_t^1 | b_{l-1}^1, z_{l-1})$, note that by the induction hypothesis,

\[
\frac{\partial \sigma_{s,t}^1}{\partial a_t^1} = \sum_{k=0}^{s-t-1} \sum_{t<\tau_1<\ldots<\tau_k<s} \left| \frac{\partial \sigma_{s,k}^1}{\partial a_{\tau_1}^1} \right| \left| \frac{\partial \sigma_{s,k}^1}{\partial a_{\tau_2}^1} \right| \ldots \left| \frac{\partial \sigma_{s,k}^1}{\partial a_{\tau_k}^1} \right| \left| \frac{\partial \sigma_{s,t}^1}{\partial a_t^1} \right|
\]

(31)

It hence follows from (25) that

\[
\frac{\partial \varphi_1}{\partial a_t^1}(a_t^1 | b_{l-1}^1, z_{l-1}) \\
\leq -c''_t(a_t^1) + \epsilon\{2^{t-1} + \sum_{s=t+1}^{T} 2^{s-t-1}\} + \epsilon\{1 + T - t\} \{1 + \epsilon \sum_{s=t+1}^{T-1} 2^{k-t-1}\}
\]

\[
= -c''_t(a_t^1) + \epsilon(2^{t-1} + 2^{T-t-1}) + \epsilon(1 + T - t)(1 + \epsilon 2^{T-t-1} - \epsilon)
\]

\[
\leq -c''_t(a_t^1) + \epsilon(\kappa + 2^{T-t} - 1) + \kappa(1 + \kappa)
\]

\[
\leq -c''_t(a_t^1) + 3\kappa < 0.
\]

This, along with the fact that

$\varphi_1(a_t^1 | b_{l-1}^1, z_{l-1}) > 0$ and $\varphi_1(a_t^1 | b_{l-1}^1, z_{l-1}) < 0$ for $a_t^1$ large enough,
implies that there exists a unique \( a_t^1 \) for which \( \varphi_1^1(a_t^1 \mid b_{t-1}, z_{t-1}) = 0 \). Define \( \sigma_t^1(b_{t-1}, z_{t-1}) \) to be this solution.

For agent 2, for each \( b_{t-1}^2 = (a_1^2, \ldots, a_{t-1}^2) \), and \( z_{t-1} \in Z_{t-1}(f) \), let \( \sigma_t^2(b_{t-1}^2, z_{t-1}) \) be the unique solution to \( \varphi_2^2(a_t^2 \mid b_{t-1}^2, z_{t-1}) = 0 \), where

\[
\varphi_2^2(a_t^2 \mid b_{t-1}^2, z_{t-1}) = -c_t'(a_t^2) + \int_{T-1} R \left\{-\Phi_T \left( \alpha_T(z_{T-1}) + a_t^2 - \Delta_{T-1} \right) - \sum_{s=t+1}^T c_t \left( \sigma_s^2((b_{t-1}^2, a_t^2), z_{s-1}) \right) \right\}
\]

\[
\times \phi_t'(x_t - \alpha_t(z_{t-1}) + a_t^2) \times \prod_{s=t+1}^{T-1} \phi_s \left( x_s - \alpha_s(z_{s-1}) + \sigma_s^2((b_{t-1}^2, a_t^2), z_{s-1}) \right) \times g_{t-1}^a(\omega_{t-1} \mid b_{t-1}^2, z_{t-1}) d\omega_{T-1}.
\]

(32)

To see that \( \sigma_t^1 \) satisfies (i), note that (30) and (10) together imply

\[
\left| c_t' \left( \sigma_t^1(b_{t-1}, z_{t-1}) \right) \right| \leq \epsilon (1 + T - t) \leq \epsilon T < 1.
\]

This further implies that \( (c_t')^{-1}(-\epsilon T) < \sigma_t^1(b_{t-1}, z_{t-1}) < (c_t')^{-1}(\epsilon T) \). Hence,

\[
c_t \left( \sigma_t^1(b_{t-1}, z_{t-1}) \right) < \max \left\{ c_t \left( (c_t')^{-1}(-\epsilon T) \right) , c_t \left( (c_t')^{-1}(\epsilon T) \right) \right\} < 1.
\]

For (ii), since \( \frac{\partial \varphi_2^1}{\partial a_t^2}(a_t^1 \mid b_{t-1}, z_{t-1}) < 0 \) as shown above, \( \sigma_t^1 \) is differentiable as a function of \( a_t^u \) \((u = 1, \ldots, t-1)\) by the implicit function theorem. Furthermore,
differentiation of \( \varphi^1_t \) with respect to \( a^1_u \) yields

\[
\frac{\partial \varphi^1_t}{\partial a^1_u}(a^1_t | b^1_{t-1}, z_{t-1})
\]

\[
= \int_{R^{t-1}} \left\{ \phi_T(\sigma^1_{t,t} - \alpha_T + \Delta T_{t-1}) \frac{\partial \sigma^1_{t,t}}{\partial a^1_u} - \sum_{s=t+1}^{T} c'_s(\sigma^1_{s,t}) \frac{\partial \sigma^1_{s,t}}{\partial a^1_u} \right\} \\
\times \phi'_s(x_s - a^1_t + \alpha_t) \prod_{s=t+1}^{T-1} \phi_s(x_s - \sigma^1_{s,t} + \alpha_s) g^\alpha_{t-1}(\omega_{t-1} | b^1_{t-1}, z_{t-1}) d\omega_{T-1}
\]

\[
+ \int_{R^{t-1}} \left\{ \Phi_T(\sigma^1_{t,t} - \alpha_T + \Delta T_{t-1}) - \sum_{s=t+1}^{T} c_s(\sigma^1_{s,t}) \right\} \\
\times \left\{ \frac{\partial g^\alpha_{t-1}}{\partial a^1_u}(\omega_{t-1} | b^1_{t-1}, z_{t-1}) - g^\alpha_{t-1}(\omega_{t-1} | b^1_{t-1}, z_{t-1}) \sum_{k=t+1}^{T-1} \phi'_k(x_k - \sigma^1_{k,t} + \alpha_k) \frac{\partial \sigma^1_{k,t}}{\partial a^1_u} \right\} \\
\times \phi'_s(x_s - a^1_t + \alpha_t) \prod_{s=t+1}^{T-1} \phi_s(x_s - \sigma^1_{s,t} + \alpha_s) d\omega_{T-1}.
\]

By (10), (25), (26), (31), and the induction hypothesis, \( \frac{\partial \varphi^1_t}{\partial a^1_u} \) can be evaluated as:

\[
\left| \frac{\partial \varphi^1_t}{\partial a^1_u}(a^1_t | b^1_{t-1}, z_{t-1}) \right|
\leq \{\epsilon 2^{t-1} + \sum_{s=t+1}^{T} 2^{s-t-1}\} \epsilon + (1 + T - t)\epsilon^2 \sum_{k=t+1}^{T-1} 2^{k-t-1} + 2\epsilon(1 + T - t)\epsilon^{T-t}
\leq \epsilon(2^{T-t-1} + 2^{T-t-1}) + \epsilon^2(1 + T - t)(2^{T-t-1} - 1) + 2\epsilon(1 + T - t)
\leq \epsilon(\kappa + 2^{T-t-1}) + \kappa\epsilon 2^{T-t-1} + 2\kappa
\leq \kappa + \kappa^2 + 2\kappa
\leq 4\kappa.
\]

Therefore, the derivative \( \frac{\partial \sigma^1_t}{\partial a^1_u} \) satisfies

\[
\left| \frac{\partial \sigma^1_t}{\partial a^1_u}(b^1_{t-1}, z_{t-1}) \right| = \left| \frac{\partial \varphi^1_t}{\partial a^1_u}(\sigma^1_t(b^1_{t-1}, z_{t-1}) | b^1_{t-1}, z_{t-1}) \right| \leq \frac{4\kappa}{c^\alpha_T(a^1_t) - 3\kappa} \leq 1.
\]

This advances the induction step and the desired conclusion follows.

Step 2. We now show that the effort choice \( a^1_t = \sigma^1_t(b^1_{t-1}, z_{t-1}) \) in any symmetric PBE \( \sigma \) must satisfy \( \varphi^1_t(a^1_t | b^1_{t-1}, z_{t-1}) = 0 \) for any \( (b^1_{t-1}, z_{t-1}) \) and \( t \).
Fix any symmetric PBE $\sigma$ and recall that $\pi_i^t(a_i^t \mid \sigma, b_{-1}^t, z_{t-1})$ denotes agent $i$'s payoff over stages $t, \ldots, T$ when (i) $i$'s history in stages $1, \ldots, t-1$ equals $h_{-1}^t = (b_{-1}^t, z_{t-1})$, (ii) $i$ takes action $a_i^t$ in stage $t$ and plays according to $\sigma_{s}^i$ in stages $s = t+1, \ldots, T$ (given $h_{-1}^t$ and $a_i^t$), and (iii) $j$ plays according to $\sigma_{j}^i$ in every stage $s$. Write $\pi_i^t(a_i^t \mid b_{-1}^t, z_{t-1}) = \pi_i^t(a_i^t \mid \sigma, b_{-1}^t, z_{t-1})$ for simplicity. Recall also that $g_{t-1}^\sigma(\omega_{t-1} \mid b_{-1}^t, z_{t-1})$ is defined as the density of $\omega_{t-1}$ conditional on $i$'s history $(b_{-1}^t, z_{t-1})$ provided that agent $j$ played according to $\sigma_{j}^i$ in stage $s = 1, \ldots, t-1$. For agent 1, $\pi_1^t(a_1^t \mid b_{-1}^t, z_{t-1})$ can be expressed as

$$
\pi_1^t(a_1^t \mid b_{-1}^t, z_{t-1}) = -c_1(a_1^t)
$$

$$
+ \int_{\mathbb{R}^{t-1}} \left\{ P(\zeta_T + \sigma_{T, t}^1(b_{t-1}^1, a_1^1, z_{t-1}) - \sigma_{T, 0}^2(z_{t-1}) > -\Delta_{T-1}) 
- \sum_{s=t+1}^{T} c_s(\sigma_{s,t}^1(b_{s-1}^1, a_1^1, z_{s-1})) \right\}
$$

$$
\times \phi_t(x_t - a_1^1 + \sigma_{T, 0}^2(z_{t-1}))
$$

$$
\times \prod_{s=t+1}^{T-1} \phi_s(x_s - \sigma_{s,t}^1(b_{s-1}^1, a_1^1, z_{s-1}) + \sigma_{s, 0}^2(z_{s-1}))
$$

$$
\times g_{t-1}^\sigma(\omega_{t-1} \mid b_{1}^t, z_{t-1}) d\omega_{T-1},
$$

where $z_s = (z_{t-1}, f_t(\omega_t), \ldots, f_s(\omega_s))$ for $s = t, \ldots, T-1$. Suppose first that $t = T$.

In this case, $\pi_T^T$ can be expressed as

$$
\pi_T^T(a_T^1 \mid b_{T-1}^t, z_{T-1})
$$

$$
= -c_T(a_T^1)
$$

$$
+ \int_{\mathbb{R}^{T-1}} \Phi_T(a_T^1 - \sigma_{T, 0}^1(z_{T-1}) + \Delta_{T-1}) g_{T-1}^\sigma(\omega_{T-1} \mid b_{1}^T, z_{T-1}) d\omega_{T-1}.
$$

Differentiating $\pi_T^T$ with respect to $a_T^1$, we obtain

$$
\frac{\partial \pi_T^T}{\partial a_T^1}(a_T^1 \mid b_{T-1}^t, z_{T-1})
$$

$$
= -c_T'(a_T^1)
$$

$$
+ \int_{\mathbb{R}^{T-1}} \phi_T(a_T^1 - \sigma_{T, 0}^1(z_{T-1}) + \Delta_{T-1}) g_{T-1}^\sigma(\omega_{T-1} \mid b_{1}^T, z_{T-1}) d\omega_{T-1}.
$$

Sequential rationality of $\sigma_T^T$ implies

$$
c_T'(\sigma_T^T(b_{T-1}^T, z_{T-1})) = \int_{\mathbb{R}^{T-1}} \phi_T(\sigma_T^1(b_{T-1}^1, z_{T-1}) - \sigma_{T, 0}^2(z_{T-1}) + \Delta_{T-1})
$$

$$
\times g_{T-1}^\sigma(\omega_{T-1} \mid b_{1}^T, z_{T-1}) d\omega_{T-1}.
$$
The corresponding FOC for agent 2 is given by

\[ c_T'(\sigma^2_T(b^2_T, z_{T-1})) = \int_{R_{T-1}} \Phi_T \left( -\sigma^1_T(z_{T-1}) + \sigma^2_T(b^2_T, z_{T-1}) - \Delta_T \right) \times g_{T-1}^\sigma(\omega_{T-1} | b^2_T, z_{T-1}) d\omega_{T-1}. \]

When \( b^0_T \) equals the action sequence induced by \( \sigma^i \) along \( z_{T-1} \), we have

\[ \sigma^1_T(b^0_T, z_{T-1}) = \sigma^1_T(z_{T-1}), \quad \text{and} \]

\[ g^\sigma_T(\omega_T | b^0_T, z_{T-1}) = g_T(\omega_T | z_{T-1}) \]

by definition so that (34) and (35) imply that the stage \( T \) effort on the equilibrium path should satisfy

\[ \sigma^1_T(z_{T-1}) = \sigma^2_T(z_{T-1}) = (c_T')^{-1} \left( E^{\sigma,f}[\Phi_T(\tilde{\Delta}_T - z_{T-1})] \right) = \alpha_T(z_{T-1}). \]

It follows from (36) that (30) and (32) are equivalent to \( \varphi^1_T(a^1_T | b^1_T, z_{T-1}) = 0 \) and \( \varphi^2_T(a^0_T | b^0_T, z_{T-1}) = 0 \), respectively.

As an induction hypothesis, fix \( t < T \) and suppose that the FOC for agent \( i \)'s payoff maximization in stage \( s \) is given by \( \varphi^i_s(a^i_s | b^i_s, z_s) = 0 \) (s = t + 1, . . . , T). By Step 1, \( \sigma^i_s \) (s = t + 1, . . . , T) (defined in (28) and (29)) is differentiable as a function of \( a^i_t \), and hence so is \( \pi^i_t(\cdot | b^i_{t-1}, z_{t-1}) \). Using the envelope theorem, we can differentiate (33) to obtain

\[
\frac{\partial \pi^i_t}{\partial a^i_t}(a^i_t | b^i_{t-1}, z_{t-1}) = -c'_t(a^i_t) \]

\[
- \int_{R_{T-1}} \left\{ \Phi_{T,t} \left( \sigma^1_{T,t}(b^1_t, a^i_t, z_{t-1}) - \sigma^2_{T,0}(z_{T-1}) - \Delta_T \right) \right. \]

\[
- \sum_{s=t+1}^{T} c_s \left( \sigma^1_{s,t}(b^1_s, a^i_t, z_s) \right) \left. \right\} \]

\[
\times \phi_t(x_t - a^i_t + \sigma^2_{t,0}(z_{t-1})) \]

\[
\times \prod_{s=t+1}^{T-1} \phi_s(x_s - \sigma^1_{s,t}(b^1_s, a^i_t, z_s) + \sigma^2_{s,0}(z_{s-1})) \]

\[
\times g^\sigma_{t-1}(\omega_{t-1} | b^1_{t-1}, z_{t-1}) d\omega_{T-1}. \]

47
Sequential rationality of $\sigma^1_t$ implies

$$c'_t(\sigma^1_t(b^1_{t-1}, z_{t-1}))$$

$$= - \int_{R^{T-1}} \left\{ \Phi_T \left( -\sigma^1_{T,t-1}(b^1_{t-1}, z_T) + \sigma^2_{T,0}(z_{T-1}) + \Delta_{T-1} \right) - \sum_{s=t+1}^{T} c_s(\sigma^1_{s,t-1}(b^1_{t-1}, z_{s-1})) \right\} \times \phi'_t \left( x_t - \sigma^1_{t,0}(z_{t-1}) + \sigma^2_{t,0}(z_{t-1}) \right)$$

$$\times \prod_{s=t+1}^{T-1} \phi_s \left( x_s - \sigma^1_{s,t-1}(b^1_{t-1}, z_{s-1}) + \sigma^2_{s,0}(z_{s-1}) \right)$$

$$\times g^\sigma_{t-1}(\omega_{t-1} | b^1_{t-1}, z_{t-1}) d\omega_{T-1}. \quad (37)$$

The corresponding FOC for agent 2 is given by

$$c'_t(\sigma^1_t(b^1_{t-1}, z_{t-1}))$$

$$= - \int_{R^{T-1}} \left\{ \Phi_T \left( -\sigma^1_{1,0}(z_{T-1}) + \sigma^2_{1,t-1}(b^2_{t-1}, z_T) - \Delta_{T-1} \right) - \sum_{s=t+1}^{T} c_s(\sigma^2_{s,t-1}(b^2_{t-1}, z_{s-1})) \right\} \times \phi'_t \left( x_t - \sigma^1_{t,0}(z_{t-1}) + \sigma^2_{t,0}(z_{t-1}) \right)$$

$$\times \prod_{s=t+1}^{T-1} \phi_s \left( x_s - \sigma^1_{s,t-1}(z_{s-1}) + \sigma^2_{s,t-1}(b^2_{t-1}, z_{s-1}) \right)$$

$$\times g^\sigma_{t-1}(\omega_{t-1} | b^2_{t-1}, z_{t-1}) d\omega_{T-1}. \quad (38)$$

When $b^1_{t-1}$ equals the action sequence induced by $\sigma^1$ along $z_{t-1}$, $\sigma^1_{s,t-1}(b^1_{t-1}, z^1_{s-1}) = \sigma^1_{s,0}(z_{s-1}) (s = t, \ldots, T)$. Substituting this and the symmetry condition $\sigma^1_{s,0}(z_{s-1}) = ...$
\( \sigma^2_{s,0}(z_{s-1}) \) for each \( s \) into (36), we obtain
\[
c_t'(\sigma^1_{s,0}(z_{t-1})) = - \int_{R^{T-1}} \Phi_T(\Delta_{T-1}) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
+ \int_{R^{T-1}} \sum_{s=t}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1} = E^{\sigma,f} \phi_T(\tilde{\Delta}_{T-1}) \mid z_{t-1}
\]
where the second equality follows from integration by parts over \( x_t \). When the feedback policy \( f \) is even, we have
\[
\int_{R^{T-1}} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1} = 0.
\]
To see this, for each subset \( J \) of \( \{2, \ldots, T-1\} \), let
\[
B(J) = \{ \omega_{T-1} \in R^{T-1} : x_1 > 0, x_s > 0 \text{ if } s \in J \text{ and } x_s < 0 \text{ if } s \notin J \}.
\]
For example, \( B(J) = R_+ \times (-R_+)^{T-2} \) for \( J = \phi \) and \( B(J) = R_+^{T-1} \) for \( J = \{2, \ldots, T-1\} \). It can be seen that for each \( J \subset \{2, \ldots, T-1\} \),
\[
\int_{B(J)} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
+ \int_{-B(J)} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
= \int_{B(J)} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
- \int_{-B(J)} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(-x_t) \prod_{s=t+1}^{T-1} \phi_s(-x_s) g^f_{T-1}(-\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
= \int_{B(J)} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
- \int_{-B(J)} \sum_{s=t+1}^{T} c_s(\sigma^1_{s,0}(z_{s-1})) \phi_t'(x_t) \prod_{s=t+1}^{T-1} \phi_s(x_s) g^f_{T-1}(\omega_{t-1} \mid z_{t-1}) \, d\omega_{T-1}
\]
\[
= 0.
\]
where the first equality uses $\phi_s(x_s) = \phi_s(-x_s)$ and $\phi'_s(x_t) = -\phi'_s(-x_t)$, and the second uses the change of variables and the fact that $z_{s-1} = (f_1(\omega_1), \ldots, f_{s-1}(\omega_{s-1})) = (f_1(-\omega_1), \ldots, f_{s-1}(-\omega_{s-1}))$. (40) follows if we note

$$
\int_{\mathbb{R}^{T-1}} = \sum_{J \subset \{2, \ldots, T-1\}} \left\{ \int_{B(J)} + \int_{-B(J)} \right\}.
$$

From (39) and (40), we see that the stage $t$ effort on the symmetric equilibrium path should satisfy

$$
\sigma^1_{t,0}(z_{t-1}) = \sigma^2_{t,0}(z_{t-1}) = (c'_t)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\bar{\Delta}_{T-1}) \mid z_{t-1} \right] \right) = \alpha_t(z_{t-1}).
$$

(41) shows that (37) and (38) are equivalent to $\varphi^1_t(a^1_t \mid b^1_{t-1}, z_{t-1}) = 0$ and $\varphi^2_t(a^2_t \mid b^2_{t-1}, z_{t-1}) = 0$, respectively.

Step 3. Finally, we show that the effort choice defined by $\varphi^1_t(a^1_t \mid b^1_{t-1}, z_{t-1}) = 0$ maximizes each agent’s payoff. For this, it suffices to verify that we have from Steps 1 and 2,

$$
\frac{\partial^2 \pi^1_t}{\partial (a^1_t)^2} (a^1_t \mid b^1_{t-1}, z_{t-1}) = \frac{\partial \varphi^1_t}{\partial a^1_t} (a^1_t \mid b^1_{t-1}, z_{t-1}) < 0
$$

for any $a^1_t$ and $t$. //

**Proof of Theorem 15** Take any feedback policy $f$ that admits a symmetric pure PBE $\sigma$. Since $(c'_t)^{-1}$ is concave, Jensen’s inequality implies that

$$
E^{\sigma,f} \left[ \sigma^1_t(\bar{z}_{t-1}) \right] = E^{\sigma,f} \left[ (c'_t)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\bar{\Delta}_{T-1}) \mid \bar{z}_{t-1} \right] \right) \right] \\
= E^{\sigma,f} \left[ (c'_t)^{-1} \left( E^{\sigma,f} \left[ E^{\sigma,f} \left[ \phi_T(\bar{\Delta}_{T-1}) \mid \bar{\omega}_{t-1} \right] \mid \bar{z}_{t-1} \right] \right) \right] \\
\leq E^{\sigma,f} \left[ (c'_t)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\bar{\Delta}_{T-1}) \mid \bar{\omega}_{t-1} \right] \mid \bar{z}_{t-1} \right) \right] \\
= E^{\sigma,f} \left[ (c'_t)^{-1} \left( E^{\sigma,f} \left[ \phi_T(\bar{\Delta}_{T-1}) \mid \bar{\omega}_{t-1} \right] \right) \right].
$$

On the other hand,

$$
E^{\sigma,f} \left[ \phi_T(\bar{\Delta}_{T-1}) \mid \bar{\omega}_{t-1} \right] = \int_{\mathbb{R}^{T-1}} \phi_T(\Delta_{t-1} + \sum_{s=t}^{T-1} x_s) \prod_{s=t}^{T-1} \phi_s(x_s) \, dx_t \cdots dx_{T-1} \\
= (\phi_t \ast \cdots \ast \phi_T)(\Delta_{t-1}).
$$
Substituting this into the above, we obtain

\[ E^{\sigma,f}_{\sigma}(\tilde{z}_{t-1}) \leq E^{\sigma,f}_{\sigma}(\tilde{c}'_{t-1}((\phi_t * \cdots * \phi_T)(\tilde{\Delta}_{t-1}))) \]

\[ = \int_{R^{t-1}} (\tilde{c}'_{t-1}((\phi_t * \cdots * \phi_T)(|\Delta_{t-1}|)) \prod_{s=1}^{t-1} \phi_s(x_s) \, d\omega_{t-1}. \]

Since the far right-hand side equals the expected stage \( t \) effort in the symmetric PBE under the feedback policy in (12), the desired conclusion follows. //

References


