Coordinating Adoption Decisions under Externalities and Incomplete Information

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Abstract

A monopolist sells a good whose value depends on the number of buyers who adopt it as well as on their private types. The seller coordinates the buyers' adoption decisions based on their reported types, and charges them the price based on the number of adoptions. We study ex post implementable sales schemes that are collusion-proof, and show that under the revenue maximizing scheme, more buyer types are willing to adopt when there are more adoptions, and the number of adoptions is maximized subject to the participation constraints.

Key words: network externalities, strategy-proofness, revenue maximization, coalition, collusion, user group.

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1 Introduction

Goods have adoption externalities when their value to any consumer depends on the consumption decision of other consumers. A classical example of a good with adoption externalities is a telecommunication device whose value depends directly on the number of other people using the device. Other leading examples include the operating system (OS) of PC's, fuel-cell vehicles, social networking services, industrial parks, and so on. The nature of externalities may be purely physical as in the case of the telecommunication device, but may also be market-based or

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psychological. Market-based externalities arise when more users of a good induces the market to provide complementary goods that enhance the value of the good. More users of a fuel-cell vehicle, for example, encourages entry into the market of charge stations, which leads to the increased value of such vehicles. On the other hand, much of bandwagon consumption in the fashion, toy and electronic industries is best explained by psychological externalities where consumers' tastes for a particular good are directly influenced by the size of its consumption. When all types of externalities are considered, it would be no exaggeration to say that a substantial fraction of goods have such a property.

Despite their importance, goods with adoption externalities have received relatively little attention in economic theory.[1] Analysis in the literature has mostly focused on the resolution of the coordination problem arising from the multiplicity of equilibria. When every consumer expects others to adopt the good, its expected value is high enough to render adoption a rational decision (at least for some price). On the other hand, when every consumer expects no other consumers to adopt, then its low expected value makes no adoption rational. Expectation is self-fulfilling in both cases, leading to multiple, Pareto-ranked equilibria. A subsidy scheme as proposed by Dybvig and Spatt (1983) is one way to eliminate the problem by promising to compensate the adopters when the number of adoptions is below some threshold. The existence of Pareto-ranked equilibria is also the main focus of the analysis of intertemporal adoption decisions.[2] In contrast, the problem of revenue maximization by a monopolist has been analyzed only indirectly either under the implicit assumption that higher participation implies higher revenue, or through the analysis of introductory prices, a common practice of setting a low price for early adopters and a higher, regular price for others (Cabral *et al.*, 1999).[3] The objective of this paper is to directly explore the revenue maximizing coordination and pricing of a good with adoption externalities under incomplete information.

We suppose that there are N ex ante symmetric buyers who choose whether to adopt the seller's good or not. A *(user) group* is the set of adopting buyers. Each buyer i's valuation v_i of the good is an increasing function of the size of the user

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[1]Rohlfs (1974) is the first to give a theoretical analysis of goods with externalities.

[2]See Gale (1995, 2001), Ochs and Park (2010) and Shichijo and Nakayama (2009).

[3]Sekiguchi (2009) examines the monopolist's revenue in the dynamic setup as in Gale (1995) when the price is held constant over time and across consumers. Aoyagi (2010) analyzes a related but different problem in which a monopolist attempts to maximize revenue when the buyers' valuations mutually depend on one another's types.

group and his private type which is distributed over the unit interval. A *coordinating scheme* determines a user group as a function of the buyers' reported types, and determines transfer from each buyer as a function of the size of the realized user group.

We envisage the situation where the buyers know each other well and collusion among them is a plausible concern for the seller as in the case of the sale of an intermediate good where the buyers come from the same industry. Specifically, co-ordinating schemes are required to be not only strategy-proof, but also coalitionally strategy-proof. Coalitional strategy-proofness is a strengthening of strategy proof-ness, and ensures that at any type profile, no subset of buyers can benefit from jointly misreporting their types.

Our analysis highlights one simple property of a coordinating scheme named monotonicity. Given the price of each group, consider the marginal type of a buyer who is just indifferent between adopting as part of the size n user group for price t^n and not adopting. We say that a size m user group priced at t^m is *more accessible* than a size n user group priced at t^n if, whenever the marginal type for the size m group is lower than the marginal type for the size n group.[4] In other words, one group is more accessible than another if any buyer type who accepts to be part of the first group accepts to be part of the second group. We say that a coordinating scheme is *monotone* if (1) a larger user group is always more accessible than a smaller user group, and (2) the largest user group is chosen as permitted by individual rationality. The latter property implies that a monotone scheme is efficient in the sense that it does not exclude any buyer type who is willing to adopt given the price and the decisions of other buyers.[5] We show that a monotone coordinating scheme is coalitionally implementable, and establish as the main theorem of this paper that a coordinating scheme is monotone if it is optimal in the class of coalitionally implementable schemes.

The idea of a coordinating scheme generalizes an inducement scheme proposed by Park (2004). An inducement scheme, which itself generalizes the subsidy schemes discussed above to the incomplete information environment, is a sales mechanism in which the transfer between the seller and buyers depends on the realized user group. It first posts a price of each user group, and then lets the buyers simultane-ously decide whether to adopt or not. Because of this feature, the buyers' adoption

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[4]Note that this does not imply that $t^m < t^n$ since the values of the two groups are different.

[5]In fact, it will be shown that a symmetric coordinating scheme is efficient in this sense and satisfies individual rationality only if it is monotone.

decisions are independent of one another under an inducement scheme. In contrast, we model a seller who actively coordinates adoption decisions, and propose a sales scheme that works as a coordinating device.

The perceived multiplicity of equilibria in problems with adoption externalities makes (coalitional) strategy-proofness a preferable incentive condition compared with Bayesian incentive compatibility. One unique aspect of the present analysis is that it combines (coalitional) strategy-proofness, which is independent of the type distribution, and revenue maximization, which requires the specification of the distribution.[6] The optimality of a monotone scheme holds for any type distribution, and hence is *distribution-free.*

In line with the existing research on adoption externalities, we suppose that pricing is *adoption-contingent* in the sense that a single price is associated with each possible group. Adoption-contingent pricing under externalities is extensively analyzed in various contracting problems where the principal maximizes the revenue or minimizes the cost.[7] Introduction of incomplete information about buyer types distinguishes our model from the the existing models of adoption-contingent pricing.

As a result of revenue maximization, only a subset of buyers may end up consuming the good. A similar framework is found in the problem of excludable public goods where the planner can exclude some agents from consumption. However, the public good literature typically assumes that the good's value depends on the amount of contributions from the agents rather than their adoption status, and focuses on the efficient cost sharing rather than revenue maximization.[8]

The paper is organized as follows: The next section introduces a coordinating scheme. Ex post implementable schemes are characterized in Section 3. We demonstrate the optimality of a monotone coordinating scheme in Section 4, and conclude in Section 5. All the proofs are collected in the Appendix.

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[6]A similar approach is taken by Shao and Zhou (2008), who combine strategy-proofness and expected surplus maximization in an allocation problem of an indivisible good to two buyers. One interpretation is that the buyers have common knowledge about one another's type, but the seller only knows their distribution.

[7]See Armstrong (2006), Bernstein and Winter (2010), and Segal (2003), among others.

[8]See, for example, Moulin (1994), Deb and Razzolini (1999a, b), and Bag and Winter (1999).

2 Model

There is a good with N potential buyers indexed by $i \in I = \{1, \ldots, N\}$ each of whom either adopts the good, or not. A *(user) group* is the set of adopting buyers. A buyer's utility, denoted $u_i(G, s_i)$, is a function of the realized user group G as well as his own type s_i, which is independent and identically distributed with strictly positive density over the unit interval $S_i = [0, 1]$. We suppose that there exist functions $v^1, \ldots, v^N : [0, 1] \to \boldsymbol{R}_+$ such that

$$u_i(G, s_i) = \begin{cases} v^{|G|}(s_i) & \text{if } i \in G, \\ 0 & \text{otherwise,} \end{cases}$$

where $|G|$ denotes the cardinality of the set G. That is, the value of the good to each adopting buyer is a function only of the size of the user group, and the value to each no-adopting buyer is normalized to zero regardless of his type. We make the following assumption on $v^1, \ldots, v^N$.

Assumption 1 *For any* $n = 1, \ldots, N$,

1) $v^n(0) = 0$,

2) $(v^n)'(\cdot) > 0$,

3) If $m < n$, then $(v^m)'(\cdot) < (v^n)'(\cdot)$.

That is, (1) the value of the good equals zero to a buyer of the lowest type $s_i = 0$ regardless of the user group, (2) the value is strictly increasing with the private type, and (3) the larger the user group, the larger the marginal impact of the private type on the value.

A *coordinating scheme* determines the user group as a function of the private type profile, and the monetary transfer from each buyer as a function of the realized group. Let $S = \prod_{i \in I} S_i$ be the set of type profiles $s = (s_i)_{i \in I}$. Formally, a coordinating scheme is a pair (f, x) of an *assignment rule* $f : S \to 2^I$ and a *pricing rule* $x = (x_1, \ldots, x_N) : 2^I \to \boldsymbol{R}^I$: $f(s) \subset I$ is the user group formed under the type profile $s \in S$, and $x_i(G) \in \boldsymbol{R}$ is the monetary transfer from buyer i when user group $G \subset I$ is formed. When buyer i is not assigned the good under user group G

$(i \notin G)$, we set his transfer equal to zero: $x_i(G) = 0$.[9] A coordinating scheme (f, x) is *strategy-proof* if for every i, s_i, s_i' and s_{-i},

$$u_i(f(s_i, s_{-i}), s_i) - x_i(f(s_i, s_{-i})) \geq u_i(f(s_i', s_{-i}), s_i) - x_i(f(s_i', s_{-i})),$$

and *ex post individually rational* if for any i, s_i, and s_{-i},

$$u_i(f(s_i, s_{-i}), s_i) - x_i(f(s_i, s_{-i})) \geq 0.$$

A coordinating scheme (f, x) is *ex post implementable* if it is both strategy-proof and ex post individually rational. Our analysis focuses on a stronger form of strategy-proofness defined as follows: Given a coordinating scheme (f, x), a subset $J \subset I$ of buyers, and type profiles $s = (s_J, s_{-J})$ and $\hat{s}_J$, $\hat{s}_J$ is a *profitable deviation* for the coalition J at s if

$$u_i(f(\hat{s}_J, s_{-J}), s_i) - x_i(f(\hat{s}_J, s_{-J})) \geq u_i(f(s), s_i) - x_i(f(s)) \text{ for every } i \in J, \text{ and}$$
$$u_i(f(\hat{s}_J, s_{-J}), s_i) - x_i(f(\hat{s}_J, s_{-J})) > u_i(f(s), s_i) - x_i(f(s)) \text{ for some } i \in J.$$

(f, x) is *coalitionally strategy-proof* if no coalition of buyers has a profitable deviation at any type profile. A coordinating scheme with this property is hence robust against buyer collusion since even if a subset of buyers share information about their private types and jointly misreport them, the deviation is not profitable.[10] Note that the definition does not allow side-transfers among coalition members.[11] (f, x) is *coalitionally implementable* if it is coalitionally strategy-proof and ex post individually rational. Given the concern for the multiplicity of equilibria in the presence of externalities, (coalitional) strategy-proofness is a particularly suitable requirement compared with Bayesian incentive compatibility, which does not address the multiplicity issue.[12]

We say that a coordinating scheme (f, x) is *constrained efficient* if given x, no other user group achieves a higher aggregate net welfare than $f(s)$ for any profile s:

$$\sum_i \{u_i(f(s), s_i) - x_i(f(s))\} \geq \sum_i \{u_i(G, s_i) - x_i(G)\}$$

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[9]This is without loss of generality since ex post IR below requires the transfer from a non-adopting buyer to be non-positive, and the revenue maximizing seller never sets a negative price for them.

[10]There is extensive analysis of coalitional (or group) strategy-proofness since Moulin (1980).

[11]Allowing side-transfers expands the set of profitable deviations, and leaves only trivial coordinating schemes coalitionally implementable.

[12]Park (2004) presents an example where the optimal Bayesian implementable mechanism admits multiple equilibria under externalities.

for every type profile s and user group $G \subset I$.[13]

Let the seller's expected revenue per buyer under a coordinating scheme (f, x) be denoted by

$$R(f, x) = \frac{1}{N} \sum_i E_s[x_i(f(s))].$$

A coalitionally implementable coordinating scheme (f, x) is *optimal* if it maximizes the expected revenue. Given the ex ante symmetry across buyers, we assume throughout our analysis that a coordinating scheme (f, x) is *symmetric* in the following sense: For any $i \neq j$,

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\begin{array}{lll}
(1) & s_i = s_j & \Rightarrow \quad \{i, j\} \subset f(s) \text{ or } \{i, j\} \subset I \setminus f(s), \\
(2) & \{i, j\} \subset f(s) & \Rightarrow \quad x_i(f(s)) = x_j(f(s)).
\end{array}
$$

That is, (1) any buyers of the same type are treated equally when it comes to the assignment of the good,[14] and (2) all adopting buyers face the same price. When (f, x) is symmetric, the transfer from each adopting buyer is a function only of the size of the user group. That is, there exists $t = (t^1, \dots, t^N) \in \mathbf{R}^N$ such that t^n is the price of the size n user group: For any $G \subset I$ with $G \neq \emptyset$ and $i \in G$,

$$x_i(G) = t^{|G|}.$$

We denote a symmetric coordinating scheme by (f, t) in what follows.

3 Characterization of Ex Post Implementability

In this section, we present a simple characterization of ex post implementability that will later be used in the analysis of optimal schemes under coalitional implementability. Fix any $s_{-i} \in S_{-i}$ and let

$$B_i(s_{-i}) = \{|f(s_i, s_{-i})| : s_i \in S_i, i \in f(s_i, s_{-i})\}$$

be the set of sizes of possible user groups including buyer i that he can induce by changing his report when the other buyers' types are fixed at s_{-i}. Furthermore, for any n and profile $s_{-i} \in S_{-i}$, let

$$L_i(n, s_{-i}) = \text{cl} \{s_i \in S_i : |f(s_i, s_{-i})| = n, i \in f(s_i, s_{-i})\}$$

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[13]Since $v^n(\cdot) \geq 0$ for any n, an ex post efficient outcome that maximizes the sum of the buyers' and seller's payoffs is achieved when $f(\cdot) = I$ and $x_i(\cdot) = 0$.

[14]A stronger requirement is that swapping of the signals of any pair of buyers result in the swapping of their assignments while leaving the assignments of all other buyers unchanged.

be the (closure of the) set of i's types that would lead to a user group of size n when others' type profile is fixed at s_{-i}.

Now suppose that (f, t) is a symmetric coordinating scheme. Given any user group of size n, define $y^n \in [0, 1]$ to be the marginal type at which a buyer is indifferent between being part of a user group of size n for price t^n, and not adopting:

$$v^n(y^n) - t^n = 0. \tag{1}$$

Such a type y^n is unique by Assumption 1 if it exists. If $v^n(0) - t^n > 0$, then let $y^n = 0$ and if $v^n(1) - t^n < 0$, then let $y^n = 1$. Moreover, given any $1 \leq m < n \leq N$, define $y^{mn} = y^{nm} \in [0, 1]$ to be the marginal type at which a buyer is indifferent between adopting as part of a user group of size m for price t^m and adopting as part of a user group of size n for price t^n:

$$v^m(y^{mn}) - t^m = v^n(y^{mn}) - t^n. \tag{2}$$

Again, such a type y^{mn} is unique if it exists. If $v^m(0) - t^m < v^n(0) - t^n$, set $y^{mn} = 0$ and if $v^m(1) - t^m > v^n(1) - t^n$, set $y^{mn} = 1$.[15]

For each n, we may restrict attention to the price t^n such that $0 \leq t^n \leq v^n(1)$. Since there is a one-to-one correspondence between any such t^n and y^n, we will interchangeably use the profile of marginal types $y = (y^1, \ldots, y^N)$ and the pricing rule $t = (t^1, \ldots, t^N)$ in what follows.

Proposition 1 *A coordinating scheme (f, t) is ex post implementable if and only if the following holds. For any i and s_{-i}, if*

$$B_i(s_{-i}) = \{k_1, \ldots, k_n\}$$

for some $1 \leq k_1 < \cdots < k_n \leq N$, then

1) $t^{k_1} \leq \cdots \leq t^{k_n}$.

2) $y^{k_1} \leq \cdots \leq y^{k_n}$.

3) $y^{k_{m-1}k_m} \leq y^{k_m k_{m+1}}$ for $m = 1, \ldots, n$, where $y^{k_0 k_1} = y^{k_1}$ and $y^{k_n k_{n+1}} = 1$.

4) For any $m = 1, \ldots, n$, $L_i(k_m, s_{-i}) = \left[y^{k_{m-1}k_m}, y^{k_m k_{m+1}} \right]$.

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[15] In other words, a buyer of type s_i prefers n to no adoption if $s_i < y^n$ and no adoption to n if $s_i > y^n$. Likewise, the buyer prefers m to n if $s_i < y^{mn}$, and n to m if $s_i > y^{mn}$.

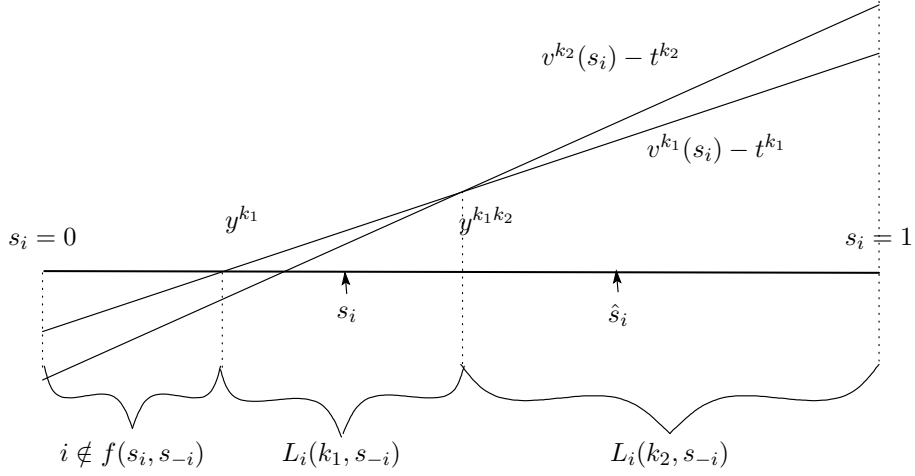Figure 1: Illustration of ex post implementability

5) $s_i < y^{k_1} \Rightarrow i \notin f(s_i, s_{-i})$.

For a series of user groups that buyer i can induce by varying his report s_i against s_{-i}, Proposition 1 states that (1) the price of each user group increases with its size,[16] (2) the marginal type for each group against no assignment increases with its size, (3) the marginal type between two neighboring group sizes is monotonically ordered, (4) each group size is associated with an interval in $[0,1]$, and (5) the left-most interval $[0, y^{k_1})$ (if non-empty) corresponds to the user groups that do not include buyer i. Note that the relative ordering between any groups of the same size is indeterminate. An ex post implementable scheme is illustrated in Figure 1: Suppose for example that i's true type is s_i as indicated in the figure. If he reports truthfully, then his payoff equals $v^{k_1}(s_i) - t^{k_1}$, which is greater than $v^{k_2}(s_i) - t^{k_2}$ that he would get by misreporting that his type is $\hat{s}_i$.

4   Optimal Schemes under Coalitional Implementability

We now turn to the characterization of optimal coordinating schemes under the requirement of coalitional implementability. We first introduce the key concept of

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[16]Note that such an ordering of payments holds only when buyer i's reports induce multiple user groups including himself against a fixed profile s_{-i}. As will be seen later, under a monotone scheme, i's reports induce at most one user group including i against a fixed s_{-i}.

monotonicity, and then establish the main result of this paper that any optimal scheme must be monotone under coalitional implementability.

We say that a user group of size n priced at t^n is *more accessible* than a user group of size m priced at t^m if $y^n \leq y^m$. In other words, whenever a buyer type is willing to adopt as part of a size m user group, that type is willing to adopt as part of a size n user group. Note that this is not equivalent to saying that the transfer required for the former group is less than that for the latter group: $t^n \leq t^m$.

Let $\lambda^0 = 1$, and for each $k = 1, \ldots, N-1$, let λ^k be the kth highest value among $N-1$ types $s_{-i} = (s_j)_{j \neq i}$. A coordinating scheme (f, t) is *monotone* if

1) $0 < y^N \leq \cdots \leq y^1 < 1$, and

2) $i \in f(s) \Leftrightarrow s_i \geq y^n$ and $\lambda^{n-1} \geq y^n$ for some $n = 1, \ldots, N$.

In a monotone coordinating scheme, hence, (1) a larger user group is more accessible than a smaller user group, and (2) the maximal user group is chosen subject to individual rationality: for any n, $|f(s)| = n$ if and only if $|\{i \in I : s_i \geq y^n\}| = n$.[17] A monotone coordinating scheme is illustrated in Figure 2 for the case $N = 2$, where the user group $f(s)$ is indicated in each region.

The following proposition shows that monotone schemes are essentially the only symmetric coordinating schemes that are constrained efficient and ex post individually rational.

Proposition 2 *Suppose that a symmetric coordinating scheme (f, t) satisfies $0 < y^1, \ldots, y^N < 1$. Then*

$$(f, t) \text{ is constrained efficient and ex post IR} \Leftrightarrow (f, t) \text{ is monotone.}$$

It also follows immediately from Proposition 1 that a monotone scheme is ex post implementable. The following proposition shows that it is in fact coalitionally implementable.

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[17]To see that a monotone scheme (f, y) has this property, note that it is clear from the definition that $|f(s)| = n$ if $|\{i \in I : s_i \geq y^n\}| = n$. For the other implication, suppose that $|f(s)| = n$. Then IR implies that $|\{i \in I : s_i \geq y^n\}| \geq n$. If the inequality is strict, then take any i such that $s_i \geq y^n$. For this i, $\lambda^n \geq y^n \geq y^{n+1}$ so that $|f(s)| \geq n+1$ must hold by definition, a contradiction.
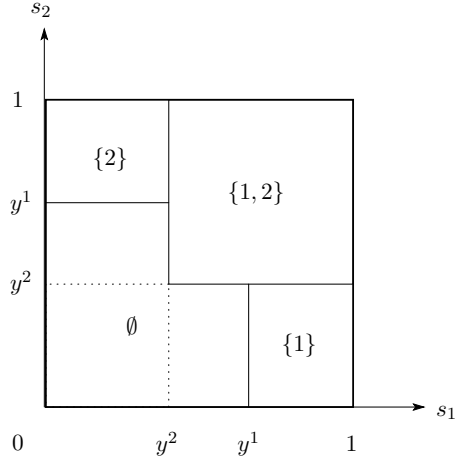
Figure 2: Monotone coordinating scheme with $N = 2$.

Proposition 3 *A monotone coordinating scheme (f, t) is coalitionally implementable.*

Figure 3 depicts a non-monotone coordinating scheme for $N = 2$ that is ex post implementable. It is ex post implementable since the assignment along each horizontal or vertical section satisfies the conditions given in Proposition 1. It is not coalitionally implementable since, for example, when (s_1, s_2) is such that $s_1 \in (y^1, y^{12})$, $s_2 \in (y^1, y^{12})$, the coalition I can claim their signals to be $\hat{s}$ such that $\hat{s}_1 > y^{12}$ and $\hat{s}_2 = s_2$, and improve from $f(s) = \emptyset$ to $f(\hat{s}) = \{2\}$. This example suggests that ex post implementability alone does not rule out complex assignment patterns.[18]

We now turn to the analysis of the seller's revenue from a coordinating scheme. Denote by $r^k(y^k)$ the expected revenue from a single buyer when a user group of size k is offered to him for the price $t^k = v^k(y^k)$. It is expressed in terms of y^k as:

$$r^k(y^k) = P(s_i \geq y^k) \, v^k(y^k).$$

It is clear from Assumption 1 that if $m < n$, then $r^m(z) < r^n(z)$ for any $z \in (0, 1)$. Since $r^n(0) = r^n(1) = 0$ for any n, size n groups do not contribute to the revenue when $y^n = 0$ or 1. Define then

$$K(f) = \{n : n = 1, \ldots, N, |f(s)| = n \text{ for some } s, \text{ and } y^n \in (0, 1)\}$$

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[18]More complications arise as the number of buyers increases: There are many user groups of the same size, and ex post implementability provides no guide as to how they should be put together.
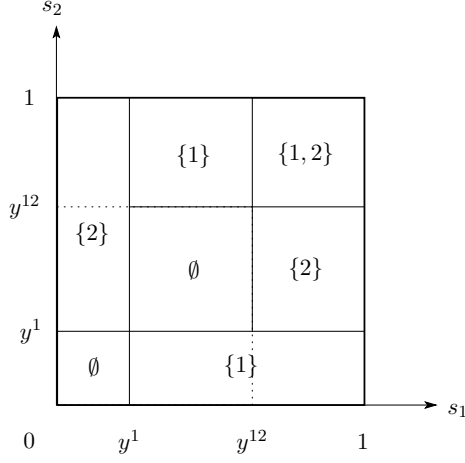
Figure 3: An ex post implementable coordinating scheme that is not monotone.

to be the set of group sizes chosen under f that contribute to the revenue. Given the marginal types $y = (y^1, \ldots, y^N)$, we further define

$$M(y) = \{m : m = 1, \ldots, N - 1, \ y^m < \max_{\ell > m} y^\ell\}.$$

$M(y)$ is the set of group sizes that are more accessible than some of the larger user groups. If (f, y) is a monotone scheme, then $y^N \leq \cdots \leq y^1$ so that $M(y) = \emptyset$, and hence $M(y) \cap K(f) = \emptyset$. The following lemma shows that this holds for any coalitionally implementable scheme.

Lemma 1 *Let (f, y) be a coalitionally implementable coordinating scheme. Then $M(y) \cap K(f) = \emptyset$.*

In other words, when size n groups are formed under a coalitionally implementable scheme (f, y), they must be less accessible than any larger user groups. In particular, if $K(f) = \{k_1, \ldots, k_n\}$ for $k_1 < \cdots < k_n$, then $y^{k_n} \leq \cdots \leq y^{k_1}$.

Recall that $R(f, y)$ denotes the seller's (per buyer) expected revenue under a coordinating scheme (f, y). We next show that when (f, y) is coalitionally implementable, $R(f, y)$ depends on f only through the set $K(f)$ of group sizes chosen by f. To this end, define for any set $K \subset \{1, \ldots, N\} = I$ of group sizes and marginal types $y = (y^1, \ldots, y^N)$ satisfying $K \cap M(y) = \emptyset$,

$$w(K, y) = \sum_{k \in K} P\left(\lambda^{k-1} \geq y^k, \ \max_{\substack{\ell > k \\ \ell \in K}} (\lambda^{\ell-1} - y^\ell) < 0\right) r^k(y^k).$$

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Intuitively, $w(K, y)$ equals the seller's expected revenue from buyer i under a coordinating scheme (f, y) when $K(f) = K$, and when f chooses the maximal group size in K subject only to (1) buyer i's incentive condition (as embodied in $r^k(y^k)$), and (2) the IR condition of all buyers: whenever there are $k - 1$ buyers with types above y^k, but there are not enough buyers with types above y^ℓ $(\ell > k)$, buyer i is offered a group of size k consisting of those $k - 1$ buyers and himself. Given that coalitional implementability imposes numerous other incentive conditions, $w(K(f), y)$ is an upper bound for $R(f, y)$. However, we can readily verify that for y such that $0 < y^I \leq \cdots \leq y^1 < 1$,

$$R(f, y) = w(I, y) \tag{3}$$

if and only if (f, y) is a monotone scheme. The following proposition shows that for any coalitionally implementable scheme (f, y), $w(K(f), y)$ gives its expected revenue $R(f, y)$.

Lemma 2 *Let (f, y) be a coalitionally implementable coordinating scheme. Then $R(f, y) = w(K(f), y)$.*

Based on Lemmas 1 and 2, the following theorem establishes the optimality of a monotone scheme by showing that the seller is better off forming user groups of all sizes than forming only some of them. Specifically, it shows that for any (K, y) such that $K \neq I$, there exists $z = (z^1, \ldots, z^N)$ such that $0 < z^N \leq \cdots \leq z^1 < 1$ and $w(K, y) < w(I, z)$. This along with (3) proves the claim.

Theorem 1 *If (f, t) is optimal in the class of coalitionally implementable coordinating schemes, then it is monotone.*

As seen in the Appendix, the proofs of Lemmas 1 and 2 only utilize deviations by the grand coalition I. Hence, Theorem 1 holds even if coalitional implementability is replaced by ex post implementability along with the requirement that no deviation by the grand coalition be profitable. This is presented as a corollary below.

Corollary 2 *If (f, t) is optimal in the class of ex post implementable coordinating schemes in which no deviation by the grand coalition I is profitable, then it is monotone.*

Although larger groups are more accessible than smaller groups under monotonicity, the following example shows that transfer required for a larger group can be higher than that required for a smaller group.

Example: Suppose that $N = 2$, and that the buyers' valuation functions are given by

$$v^1(s_i) = \gamma s_i \quad v^2(s_i) = \delta s_i,$$

where $0 < \gamma < \delta$. Since these satisfy Assumption 1, the optimal coalitionally implementable scheme (f, y) is monotone by Theorem 1, and the seller's expected revenue (per buyer) is given by

$$R(f, y) = P(s_1 \geq y^2)P(s_2 \geq y^2)v^2(y^2) + P(s_1 \geq y^1)P(s_2 < y^2)v^1(y^1)$$
$$= P(s_2 \geq y^2)r^2(y^2) + P(s_2 < y^2)r^1(y^1).$$

The optimal marginal types y^1 and y^2 solve the following problem:

$$\max_{y^1, y^2} R(f, y) \tag{4}$$
$$\text{subject to } 1 \geq y^1 \geq y^2 \geq 0.$$

Suppose now that s_i has the uniform distribution over $[0, 1]$. Ignoring the constraint for the moment, we can solve (4) for y^1 to obtain $y^1 = \frac{1}{2}$. Given this, $z = y^2$ for joint adoption solves

$$\max_z \delta z(1 - z)^2 + \frac{\gamma}{4} z,$$

which yields[19]

$$y^2 = \frac{1}{3\delta} \left\{ 2\delta - \sqrt{\delta^2 - \frac{3}{4} \gamma \delta} \right\}.$$

We can verify that $y^2 < \frac{1}{2} = y^1$ if and only if $\gamma < \delta$. Consider now the price of each user group associated with these marginal types. They are given by

$$t^1 = \frac{\gamma}{2}, \quad \text{and} \quad t^2 = \frac{1}{3} \left\{ 2\delta - \sqrt{\delta^2 - \frac{3}{4} \gamma \delta} \right\}.$$

We see that $t^1 < t^2$ if and only if

$$\frac{\delta}{\gamma} > \frac{3}{4},$$

which is true since $\delta > \gamma$. In this example, hence, the larger user group is more expensive than the smaller user group although the former is more accessible.

---

[19] As seen, analytical derivation of an optimal scheme is possible only under very limited specifications of the distribution and values.

5 Conclusion

We study the monopoly sale of a good with externalities when the seller actively coordinates the buyers' adoption decisions. Ex post implementability required in our analysis eliminates the multiplicity of equilibria, a central issue in externalities problems. We present monotonicity as a key property of the optimal coalitionally implementable scheme. In a monotone scheme, a larger user group is more accessible than a smaller user group in the sense that the set of buyer types who are willing to be part of the larger group contains that for the smaller group, and given such pricing, assignment is efficient by choosing the maximal group subject to individual rationality. Although monotonicity has no direct implication on the relative prices of user groups, it is not inconsistent with a higher price for a larger user group as seen in the example in the previous section. It remains to be seen when monotonicity as defined here implies the monotonicity in prices.

In this paper, we have only looked at externalities whose magnitude increases with the group size. It would be interesting to study the case of negative externalities, or more complex externalities based on network structure.[20] Goods with externalities are often supplied competitively as in the case of cellular phones or PC operating systems. While some aspects of such competition have been analyzed by Katz and Shapiro (1985, 1986), much remains to be understood.

Appendix

Proof of Proposition 1 (Necessity) Suppose that (f, t) is ex post implementable.

1) For any $m = 1, \ldots, n-1$, take s_i and s_i' such that $i \in f(s_i, s_{-i}) \cap f(s_i', s_{-i})$, $|f(s_i, s_{-i})| = k_m$ and $|f(s_i', s_{-i})| = k_{m+1}$. Since (f, t) is strategy-proof,

$$
\begin{aligned}
v^{k_m}(s_i) - t^{k_m} &= v^{|f(s_i, s_{-i})|}(s_i) - t^{|f(s_i, s_{-i})|} \\
&\geq v^{|f(s_i', s_{-i})|}(s_i) - t^{|f(s_i', s_{-i})|} \\
&= v^{k_{m+1}}(s_i) - t^{k_{m+1}}.
\end{aligned}
$$

Rearranging and using Assumption 1, we get

$$
t^{k_{m+1}} - t^{k_m} \geq v^{k_{m+1}}(s_i) - v^{k_m}(s_i) \geq 0.
$$

---

[20]See Sundararajan (2007) for one such formulation.

2) Suppose to the contrary that $y^{k_m} > y^{k_{m+1}}$ for some $m = 1, \ldots, n-1$. Since $k_m \in B_i(s_{-i})$, there exists s_i such that $i \in f(s_i, s_{-i})$ and $|f(s_i, s_{-i})| = k_m$. Moreover such an s_i should satisfy by ex post IR $s_i \geq y^{k_m}$. However, we would then have $s_i > y^{k_{m+1}}$ and also by Assumption 1,

$$v^{k_m}(s_i) - t^{k_m} = v^{k_m}(s_i) - v^{k_m}(y^{k_m}) < v^{k_{m+1}}(s_i) - v^{k_{m+1}}(y^{k_{m+1}}) = v^{k_{m+1}}(s_i) - t^{k_{m+1}}.$$

Since $k_{m+1} \in B_i(s_{-i})$, there exists $\hat{s}_i \neq s_i$ such that $i \in f(\hat{s}_i, s_{-i})$ and $|f(\hat{s}_i, s_{-i})| = k_{m+1}$, implying that buyer i of type s_i is strictly better off reporting $\hat{s}_i$. It follows that (f, t) is not strategy-proof, a contradiction.

3) Suppose first that $y^{k_m k_{m+1}} < y^{k_{m-1} k_m}$ for some $m = 2, \ldots, n-1$. By the definition of the marginal types, if $s_i > y^{k_m k_{m+1}}$, then type s_i strictly prefers a size k_{m+1} group to a size k_m group, and if $s_i \leq y^{k_m k_{m+1}}$, then $s_i < y^{k_{m-1} k_m}$ so that type s_i strictly prefers a size k_{m-1} group to a size k_m group. By assumption, there exist s_i, s_i' and s_i'' such that $i \in f(s_i, s_{-i}) \cap f(s_i', s_{-i}) \cap f(s_i'', s_{-i})$, $|f(s_i, s_{-i})| = k_m$, $|f(s_i', s_{-i})| = k_{m-1}$ and $|f(s_i'', s_{-i})| = k_{m+1}$. Then type s_i is strictly better off reporting s_i' if $s_i \leq y^{k_m k_{m+1}}$, and reporting s_i'' if $s_i > y^{k_m k_{m+1}}$. This is a contradiction. Suppose next that $y^{k_1 k_2} < y^{k_1}$. If $s_i \leq y^{k_1 k_2}$, then $s_i < y^{k_1}$ so that type s_i strictly prefers no assignment to a size k_1 group, and if $s_i > y^{k_1 k_2}$, then type s_i strictly prefers a size k_2 group to a size k_1 group. We then have a contradiction just as above.

4) Suppose that $s_i \in (y^{k_{m-1} k_m}, y^{k_m k_{m+1}})$ for some $m = 1, \ldots, n$. By (3) above, $s_i > y^{k_{\ell-1} k_\ell}$ for every $\ell = 1, \ldots, m$, and $s_i < y^{k_\ell k_{\ell+1}}$ for every $\ell = m, \ldots, n$. It then follows from the definition of the marginal types that type s_i strictly prefers a size k_m group to a size k_{m-1} group, $\ldots$, a size k_2 group to a size k_1 group, and a size k_1 group to no assignment. Likewise, type s_i strictly prefers a size k_m group to a size k_{m+1} group. $\ldots$, a size k_{n-1} group to a size k_n group. Hence, if $i \notin f(s_i, s_{-i})$ or $|f(s_i, s_{-i})| \neq k_m$, then we have a contradiction since he is strictly better off reporting s_i' such that $|f(s_i', s_{-i})| = k_m$ and $i \in f(s_i', s_{-i})$.

5) Suppose that $s_i \in (0, y^{k_1})$. By the same argument as above, type s_i strictly prefers no assignment to a size k_1 group, $\ldots$, a size k_{n-1} group to a size k_n group. Hence, we would have a contradiction to strategy-proofness if $i \in f(s_i, s_{-i})$.

(Sufficiency) Fix $i \in I$ and $s_{-i} \in S_{-i}$. Suppose that $B_i(s_{-i}) = \{k_1, \ldots, k_m\}$ implies (1)-(4). Suppose that $s_i \in [y^{k_{m-1} k_m}, y^{k_m k_{m+1}}]$ for some $m = 2, \ldots, n$. As in the proof of (4) in the necessity part, type s_i weakly prefers a size k_m group to a size k_{m-1} group, a size k_{m-1} group to a size k_{m-2} group, $\ldots$, a size k_1 group to no

assignment, and also weakly prefers a size k_m group to a size k_{m+1} group, $\ldots$, a size k_{n-1} group to a size k_n group. It follows that

$$s_i \in [y^{k_{m-1}k_m}, y^{k_mk_{m+1}}] \quad \Rightarrow \quad v^{k_m}(s_i) - t^{k_m} = \max\left\{0, \max_{k \in B_i(s_{-i})} v^k(s_i) - t^k\right\}.$$

Hence, buyer i of type s_i has no incentive to misreport his type and also is ensured a non-negative payoff. Likewise, type $s_i \in [0, y^{k_1})$ has no incentive to misreport. Since this is true for any i and s_{-i}, (f, t) is ex post implementable.

Proof of Proposition 2 By definition, it is clear that a monotone scheme is constrained efficient and ex post IR. For the converse implication, suppose that a coordinating scheme (f, t) is not monotone. If property (1) of the definition of monotonicity holds but property (2) fails, then it is clear that (f, t) is not constrained efficient. Hence, suppose that property (1) fails. Then there exists $m < n$ such that $y^m < y^n$. Take a type profile s such that $s_1 = \cdots = s_n = z \in (y^m, y^n)$ and if $n < N$, $s_{n+1} = \cdots = s_I < \min_k y^k$. By symmetry and ex post IR, $f(s) = \{1, \ldots, n\}$ or $\emptyset$ must hold. It follows that

$$\sum_i \{u_i(f(s), s_i) - x_i(f(s))\} = \begin{cases} n\{v^n(z) - t^n\} < 0 & \text{if } f(s) = \{1, \ldots, n\}, \\ 0 & \text{if } f(s) = \emptyset. \end{cases}$$

On the other hand, for $G = \{1, \ldots, m\}$, we have

$$\sum_i \{u_i(G, s_i) - x_i(G)\} = m\{v^m(z) - t^m\} > 0,$$

implying that (f, t) is not constrained efficient.

Proof of Proposition 3 Take any $J \subset I$, $s = (s_J, s_{-J})$ and $\hat{s}_J$. Let $\hat{s}_{-J} = s_{-J}$ and denote $\hat{s} = (\hat{s}_J, \hat{s}_{-J})$, $k = |f(s)|$, and $m = |f(\hat{s})|$. By the definition of a monotone scheme, $k = |\{i : s_i \geq y^k\}|$ and $m = |\{i : \hat{s}_i \geq y^m\}|$. If $m > k$, there exists at least one buyer $i \in J$ for whom $s_i < y^m$, $\hat{s}_i \geq y^m$ and $i \in f(\hat{s})$. Since $y^m \leq y^k$, this buyer i is not assigned the good when J reports s_J, whereas when J reports $\hat{s}_J$, he would get

$$v^{|f(\hat{s})|}(s_i) - t^{|f(\hat{s})|} = v^m(s_i) - t^m < 0.$$

This implies that $\hat{s}_J$ is not a profitable deviation for J at s. If $m < k$, take any $i \in J$ for whom $i \in f(\hat{s})$ and $s_i \geq y^m$. If there exists no such $j \in J$, then $\hat{s}$ is not

a profitable deviation for J. Since $y^m \geq y^k$, we have $v^k(y^m) - v^m(y^m) \geq v^k(y^k) - v^m(y^m)$. Furthermore, since $(v^k)' \geq (v^m)'$, $v^k(s_i) - v^m(s_i) \geq v^k(y^m) - v^m(y^m)$ for any $s_i \geq y^m$. It follows that $v^k(s_i) - v^m(s_i) \geq v^k(y^k) - v^m(y^m)$ for any $s_i \geq y^m$. In other words, if $i \in f(\hat{s})$ and $s_i \geq y^m$, then

$$\begin{aligned}
v^{|f(\hat{s})|}(s_i) - t^{|f(\hat{s})|} &= v^m(s_i) - t^m \\
&= v^m(s_i) - v^m(y^m) \\
&\leq v^k(s_i) - v^k(y^k) \\
&= v^{|f(s)|}(s_i) - t^{|f(s)|}.
\end{aligned}$$

This implies that $\hat{s}$ is not a profitable deviation for J at s.

Proof of Lemma 1 Suppose that (f, y) is such that $M(y) \cap K(f) \neq \emptyset$, and take $m \in M(y) \cap K(f)$ so that $|f(\hat{s})| = m$ for some $\hat{s}$ and $y^m < y^n$ for some $n > m$. Without loss of generality, suppose that $f(\hat{s}) = \{1, \ldots, m\}$. Take s such that $y^m < s_1 = \cdots = s_n < y^n$ and $s_{n+1} = \cdots = s_I = 0$. Symmetry and ex post IR then imply that $f(s)$ equals either $\emptyset$, $\{n+1, \ldots, N\}$, or I. We examine these three cases in turn.

- If $f(s) = \emptyset$. then $\hat{s}$ is a profitable deviation for the coalition $J = I$ at s: Buyer $i \in \{1, \ldots, m\}$ is not assigned the good when I reports s, while when I reports $\hat{s}$, he gets
$$v^{|f(\hat{s})|}(s_i) - t^{|f(\hat{s})|} = v^m(s_i) - t^m > 0,$$
where the inequality follows from the fact that $s_i > y^m$. On the other hand, buyer $i \in \{m+1, \ldots, N\}$ is not assigned the good under either s or $\hat{s}$.

- If $f(s) = \{n+1, \ldots, N\}$, then $t^{N-n} = 0$ by ex post IR and $\hat{s}$ is a profitable deviation for the coalition $J = I$ at s: Buyer $i \in \{1, \ldots, m\}$ is not assigned the good when I reports s, while when I reports $\hat{s}$, he gets
$$v^{|f(\hat{s})|}(s_i) - t^{|f(\hat{s})|} = v^m(s_i) - t^m > 0.$$
Buyer $i \in \{m+1, \ldots, n\}$ is not assigned the good when reporting either s or $\hat{s}$. Buyer $i \in \{n+1, \ldots, N\}$ gets 0 when I reports s:
$$v^{|f(s)|}(s_i) - t^{|f(s)|} = v^{N-n}(0) - t^{N-n} = 0,$$
while he is not assigned the good when I reports $\hat{s}$.

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- If $f(s) = I$, then $t^N = 0$ by ex post IR and s is a profitable deviation for the coalition $J = I$ at $\hat{s}$: For buyer $i \in \{1, \ldots, m\}$,

$$v^{|f(s)|}(\hat{s}_i) - t^{|f(s)|} = v^N(\hat{s}_i) - t^N > v^m(\hat{s}_i) - t^m = v^{|f(\hat{s})|}(\hat{s}_i) - t^{|f(\hat{s})|}.$$

  On the other hand, buyer $i \in \{m+1, \ldots, N\}$ is not assigned the good when I reports $\hat{s}$, while when I reports s, he gets

$$v^{|f(s)|}(\hat{s}_i) - t^{|f(s)|} = v^N(\hat{s}_i) - t^N \geq 0.$$

Therefore, (f, t) is not coalitionally strategy-proof.

Proof of Lemma 2 Let (f, y) be a coalitionally implementable scheme such that $K(f) = K$. Fix $k \in K$ and s_{-i} such that

$$\lambda^{k-1} > y^k, \ \max_{\substack{\ell \in K \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0, \text{ and } s_j \notin \{y^1, \ldots, y^N\} \text{ for every } j \neq i.$$

For any such s_{-i}, $|f(s_i, s_{-i})| \leq k$ for any s_i by ex post IR. Moreover, if $s_i < y^k$, then $s_i < y^m$ for any $m < k$ with $m \in K$ by Lemma 1 so that i is not assigned the good: $i \notin f(s_i, s_{-i})$. In what follows, we show that $|f(s_i, s_{-i})| = k$ for any such s_{-i} whenever $s_i > y^k$. If this holds, then

$$\begin{aligned}
E_{s_i}[t^{|f(s_i, s_{-i})|} \mid s_{-i}] &= P(s_i < y^k) \, E_{s_i}[t^{|f(s_i, s_{-i})|} \mid s_i < y^k, s_{-i}] \\
&\quad + P(s_i > y^k) \, E_{s_i}[t^{|f(s_i, s_{-i})|} \mid s_i > y^k, s_{-i}] \\
&= P(s_i > y^k) \, t^k \\
&= r^k(y^k).
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
R(f, t) &= E_s[t^{f(s)}] \\
&= E_{s_{-i}}\left[E_{s_i}[t^{|f(s_i, s_{-i})|} \mid s_{-i}] \right] \\
&= \sum_{k \in K} r^k(y^k) \, P(\lambda^{k-1} \geq y^k, \ \max_{\substack{\ell \in K \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0) \\
&= w(K, y).
\end{aligned}$$

Suppose that $s_i > y^k$ and denote $s = (s_i, s_{-i})$. We will derive a contradiction when $m = |f(s)| < k$. Let $J \subset I$ be such that $|J| = k$, and

$$f(s) \subset J \subset \{j : s_j > y^k\}.$$

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Such a set J exists since $j \in f(s)$ implies that $s_j \geq y^m$ by ex post IR, $y^m \geq y^k$ by Lemma 1, and $|\{j : s_j > y^k\}| \geq k$ and $s_j \neq y^m$ by our choice of s_{-i}. Now take $\hat{s}$ such that

$$\hat{s}_j = \begin{cases} 1 & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry, we must have either $f(\hat{s}) = J$, I, or $I \setminus J$. We examine these three in turn.

- If $f(\hat{s}) = I$, then $t^N = 0$ by ex post IR, and $\hat{s}$ is a profitable deviation for I at s: For buyer $j \in f(s)$,

$$v^{|f(s)|}(s_j) - t^{|f(s)|} = v^m(s_j) - t^m < v^N(s_j) = v^{|f(\hat{s})|}(s_j) - t^{|f(\hat{s})|}.$$

  Buyer $j \notin f(s)$ also gets $v^N(s_j)$ when I reports $\hat{s}$ while he is not assigned the good when I reports s.

- If $f(\hat{s}) = I \setminus J$, then $t^{N-k} = 0$ by ex post IR, and s is a profitable deviation for I at $\hat{s}$: Buyer $j \in f(s)$ is not assigned the good when I reports $\hat{s}$, while when I reports s, he gets

$$v^{|f(s)|}(s_j) - t^{|f(s)|} = v^m(s_j) - v^m(y^m) > 0.$$

  Buyer $j \in J \setminus f(s)$ is not assigned the good whether I reports s or $\hat{s}$. Buyer $j \in I \setminus J$ is not assigned the good when I reports s while when I reports $\hat{s}$, he also gets zero since

$$v^{|f(\hat{s})|}(s_j) - t^{|f(\hat{s})|} = v^{N-k}(0) - t^{N-k} = 0.$$

- If $f(\hat{s}) = J$, we will show that $\hat{s}$ is a profitable deviation for I at s: Note first that when $s_j > y^k$,

$$v^k(s_j) - v^m(s_j) > v^k(y^k) - v^m(y^k) \geq v^k(y^k) - v^m(y^m), \tag{5}$$

  where the first inequality follows from $(v^k)' > (v^m)'$, and the second from $y^k \leq y^m$. For buyer $j \in f(s)$, (5) implies that

$$\begin{aligned} v^{|f(\hat{s})|}(s_j) - t^{|f(\hat{s})|} &= v^k(s_j) - v^k(y^k) \\ &> v^m(s_j) - v^m(y^m) \\ &= v^{|f(s)|}(s_j) - t^{|f(s)|}. \end{aligned} \tag{6}$$

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For buyer $j \in J \setminus f(s)$, he is not assigned the good when I reports s, while when I reports $\hat{s}$, he gets

$$v^{|f(\hat{s})|}(s_j) - t^{|f(\hat{s})|} = v^k(s_j) - t^k > 0. \tag{7}$$

Buyer $j \in I \setminus J$ is not assigned the good whether I reports s or $\hat{s}$.

We can hence conclude that (f, t) is not strategy-proof.

Proof of Theorem 1 We first show that when $y^n = 0$ or 1 for some $n \in K$, then there exists $\hat{K}$ and $\hat{y}$ such that $\hat{K} \cap M(\hat{y}) = \emptyset$, $\min_{n \in \hat{K}} \hat{y}^n > 0$, $\max_{n \in \hat{K}} \hat{y}^n < 1$, and $w(\hat{K}, \hat{y}) \geq w(K, y)$. We next show that for any $K \neq I$ and y such that $K \cap M(y) = \emptyset$, $\min_{n \in K} y^n > 0$, and $\max_{n \in K} y^n < 1$, there exists $\hat{K}$ and $\hat{y}$ such that $\hat{K} \neq K$, $\hat{K} \supset K$, $\hat{K} \cap M(\hat{y}) = \emptyset$ and $w(\hat{K}, \hat{y}) > w(K, y)$. Repeating these two steps, we can conclude that $w(K, y) < w(I, \hat{y})$ for some $\hat{y}$ such that $\hat{y}^N \leq \cdots \leq \hat{y}^1$. Since $w(I, \cdot)$ is continuous over the compact set $\{y : y^N \leq \cdots \leq y^1\}$, it achieves a maximum at some $z = (z^1, \ldots, z^N)$ in this set, and this maximizer z should satisfy $0 < z^N$ and $z^1 < 1$ by the above. Let then f be an assignment rule such that (f, z) is monotone. Then (f, z) is optimal since $R(f, z) = w(I, z)$. Furthermore, any coalitionally implementable coordinating scheme $(\hat{f}, \hat{y})$ satisfying $R(\hat{f}, \hat{y}) = w(I, z)$ is monotone since then $K(\hat{f}) = I$ and $0 < \hat{y}^I \leq \cdots \leq \hat{y}^1 < 1$ must hold.

If $y^n = 1$ for some $n \in K$, let $\hat{K} = K \setminus \{n\}$ and $\hat{y} = y$. We then have $\hat{K} \cap M(\hat{y}) = \emptyset$ and $w(K, y) = w(\hat{K}, y)$. Iterating the removal of n for which $y^n = 1$, we see that there exists $\hat{K}$ and $\hat{y}$ such that $\hat{K} \cap M(\hat{y}) = \emptyset$, $w(K, y) = w(\hat{K}, y)$ and $\max_{n \in \hat{K}} \hat{y}^n < 1$. If $y^n = 0$ for every $n \in K$, then let $\hat{K} = I$ and $\hat{y}^n = 1/2$ for every $n = 1, \ldots, N$. Suppose then that $n \in K$ is such that $y^n > 0$ and $\max_{\substack{m > n \\ m \in K}} y^m = 0$. In this case, let $\hat{K} = K$ and $\hat{y}$ be such that

$$\hat{y}^m = \begin{cases} y^n & \text{if } m > n, \\ y^m & \text{if } m \leq n. \end{cases}$$

We then have $\min_{n \in K} \hat{y}^n > 0$, and

$$w(\hat{K}, \hat{y}) = w(K, y) + \sum_{\substack{m \in K \\ m > n}} P(\lambda^{m-1} \geq y^n, \max_{\substack{\ell > m \\ \ell \in K}} (\lambda^{\ell-1} - y^\ell) < 0)\, r^n(y^n)$$

$$> w(K, y).$$

Suppose now that $\min_{n \in K} y^n > 0$ and $\max_{n \in K} y^n < 1$. Let $n = \min I \setminus K$ and $\hat{K} = K \cup \{n\}$. If $n = 1$, then let $\hat{y}$ be such that $\hat{y}^1 = \max_{k \in K} y^k$ and $\hat{y}^k = y^k$ for $k > 1$. Then $\hat{K} \cap M(\hat{y}) = \emptyset$ and $w(\hat{K}, \hat{y})$ is given by

$$
\begin{aligned}
w(\hat{K}, \hat{y}) &= \sum_{k \in \hat{K}} P(\lambda^{k-1} \geq y^k, \max_{\substack{\ell \in \hat{K} \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0)\, r^k(y^k) \\
&= w(K, y) + P(\max_{\ell \in K} (\lambda^{\ell-1} - y^\ell) < 0)\, r^1(y^1) \\
&> w(K, y).
\end{aligned}
$$

If $n > 1$, then $n - 1 \in K$ and let $\hat{y}$ be such that

$$
\hat{y}^k = \begin{cases} y^k & \text{if } k \neq n, \\ y^{n-1} & \text{if } k = n. \end{cases}
$$

Since $\hat{K} \cap M(\hat{y}) = \emptyset$, $w(\hat{K}, \hat{y})$ is given by

$$
\begin{aligned}
w(\hat{K}, \hat{y}) &= \sum_{\ell \in \hat{K}} P\!\left(\lambda^{\ell-1} \geq \hat{y}^\ell, \max_{\substack{m \in \hat{K} \\ m > \ell}} (\lambda^{m-1} - \hat{y}^m) < 0\right) r^\ell(\hat{y}^\ell) \\
&= \sum_{\substack{\ell \in K \\ \ell < n}} P\!\left(\lambda^{\ell-1} \geq y^\ell, \lambda^{n-1} < y^{n-1}, \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\right) r^\ell(y^\ell) \\
&\quad + P\!\left(\lambda^{n-1} \geq y^{n-1}, \max_{\substack{m \in K \\ m > n}} (\lambda^{m-1} - y^m) < 0\right) r^n(y^{n-1}) \\
&\quad + \sum_{\substack{\ell \in K \\ \ell > n}} P\!\left(\lambda^{\ell-1} \geq y^\ell, \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\right) r^\ell(y^\ell).
\end{aligned}
$$

Noting that $\lambda^{n-2} < y^{n-1}$ implies $\lambda^{n-1} < y^{n-1}$, we can decompose the first line of the right-hand side above to rewrite $w(\hat{K}, \hat{y})$ as

$$
\begin{aligned}
w(\hat{K}, \hat{y}) &= \sum_{\substack{\ell \in K \\ \ell < n-1}} P\!\left(\lambda^{\ell-1} \geq y^\ell, \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\right) r^\ell(y^\ell) \\
&\quad + P\!\left(\lambda^{n-2} \geq y^{n-1}, \lambda^{n-1} < y^{n-1}, \max_{\substack{m \in K \\ m > n-1}} (\lambda^{m-1} - y^m) < 0\right) r^{n-1}(y^{n-1}) \\
&\quad + P\!\left(\lambda^{n-1} \geq y^{n-1}, \max_{\substack{m \in K \\ m > n}} (\lambda^{m-1} - y^m) < 0\right) r^n(y^{n-1}) \\
&\quad + \sum_{\substack{\ell \in K \\ \ell > n}} P\!\left(\lambda^{\ell-1} \geq y^\ell, \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\right) r^\ell(y^\ell).
\end{aligned}
$$

Summing the probabilities in the second and third lines above yields

$$P\left(\lambda^{n-2} \geq y^{n-1}, \ \lambda^{n-1} < y^{n-1}, \ \max_{\substack{m \in K \\ m > n-1}} (\lambda^{m-1} - y^m) < 0\right)$$
$$+ P\left(\lambda^{n-1} \geq y^{n-1}, \ \max_{\substack{m \in K \\ m > n}} (\lambda^{m-1} - y^m) < 0\right)$$
$$= P\left(\lambda^{n-2} \geq y^{n-1}, \ \max_{\substack{m \in K \\ m > n-1}} (\lambda^{m-1} - y^m) < 0\right).$$

Using this and the fact that $r^n(y^{n-1}) > r^{n-1}(y^{n-1})$, we obtain $w(\hat{K}, \hat{y}) > w(K, y)$.

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