Bertrand Competition under Network Externalities*

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Abstract

Two firms engage in price competition to attract buyers located on a network. The value of the good of either firm to any buyer depends on the number of neighbors on the network who adopt the same good. When the size of externalities increases linearly with the number of adoptions, we identify the set of price strategies that are consistent with an equilibrium in which one of the firms monopolizes the market. The set includes marginal cost pricing as well as *bipartition pricing,* which offers discounts to some buyers and charges markups to others. We show that marginal cost pricing fails to be an equilibrium under non-linear externalities but identify conditions for an equilibrium with bipartition pricing to be robust against perturbations in the externalities from linearity. The idea of bipartition pricing is then applied to the analysis of platform competition in a two-sided market under local and approximately linear externalities.

Key words: graphs, divide and conquer, price discrimination, two-sided markets, partition.
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1 Introduction

Goods have network externalities when their value to each user depends on the adoption decisions of others. Externalities are important not only for consumption goods but also for intermediate goods. As argued by Carvalho (2014), for example, modern production is an intricate network of firms. In such a network, a single supplier of inputs serves multiple downstream firms who are themselves linked with each other in the form of mutual production of final products or technology transfers. A downstream firm would find a higher value for the same inputs as used by other downstream firms that are linked to it.

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Despite their importance in reality, our understanding of network externalities is limited when those goods are supplied competitively. The objective of this paper is to study price competition in the presence of such externalities: We formulate a highly stylized model of price competition in which users located at the nodes of a network experience positive externalities when their neighbors adopt the same good or technology. In our model, two symmetric firms each supply goods or technologies that are incompatible with each other. Users of either good experience larger positive externalities when more of their neighbors in the network adopt the same good. In stage 1, the two firms post prices simultaneously. The prices can be perfectly discriminatory and negative, and are publicly observable. In stage 2, the buyers simultaneously decide which good to adopt or not to adopt either.

When no network externalities are present, it is clear that the unique subgame perfect equilibrium of this game has both firms offer c, the constant marginal cost of production, to all buyers. We find that marginal-cost pricing is consistent with an equilibrium with monopolization by one of the firms in an arbitrary network when the externalities are linear in the number of neighbors adopting the same good. In contrast with the case with no externalities, however, we show that various pricing strategies are consistent with an equilibrium under linear externalities. In effect, when both firms offer the same price vector z, it is consistent with a monopolization equilibrium if the sum of markups and markdowns it entails for any subset of buyers is less than or equal to (the factor of proportion times) the number of links they jointly have with the complementary set. The latter quantity can be interpreted as the aggregate externalities that these buyers could enjoy if they were connected with the rest of the network.

With no markup and markdown to any buyer, marginal cost pricing is clearly an equilibrium under linear externalities. When the externalities are non-linear, on the other hand, we show that marginal cost pricing is consistent with an equilibrium only when the buyer network is either a cycle or complete.[1] We also show that under non-linear externalities, there exists no equilibrium with monopolization in which every buyer is charged the same price. These observations lead us to the study of equilibrium pricing strategies that are robust against slight perturbations in the externalities from linearity. Our central focus in this investigation is a class of *bipartition pricing,* which entails price discrimination based on a binary partition of the buyer set: Discounts are offered to the buyers in one subset but markups are charged to the buyers in the other subset. Furthermore, the size of a markup or markdown to any buyer is proportional to the number of his neighbors in the other subset. We consider small perturbations in the externalities from linearity, and define equilibrium pricing strategies under linear externalities to be *robust* if there exists a non-degenerate set of approximately linear externalities under which there exists an equilibrium pricing strategy that is "close" to the original pricing

[1] A network is complete if any pair of buyers are neighbors. A complete network implies the existence of global externalities.

strategy. We show that in a large class of networks, bipartition pricing given some binary partition of buyers is in fact robust.

One important class of networks that admits a natural interpretation of robust bipartition pricing identified above is *bipartite* networks in which the buyer set is partitioned into two subsets and each buyer in one subset has neighbors only in the other subset. Bipartite networks are a graph-theoretical representation of *two-sided markets* that have received much attention in the literature: The two subsets correspond to the two sides of the market such as buyers and sellers of a certain good, and the two firms correspond to platforms that compete in offering marketplace to them. In a robust bipartition equilibrium over a bipartite network, all users on one side of the market are charged markups while all users on the other side are offered discounts. Furthermore, the size of the markup or discount to each user is proportional to the number of users on the other side of the market who are directly connected to them. This is the first result that demonstrates that this popular pricing strategy arises as an equilibrium of price competition under local network externalities, and also identifies how the markups or markdowns are related to the specification of those externalities.[2]

As mentioned above, the key assumption of our model is the ability of the firms to perfectly price discriminate the buyers. This assumption is more likely satisfied in intermediate goods markets with a limited number of participants than large consumption goods markets. One good example is provided by the international competition in the sales of infrastructures that has recently become a major form of international trade. In a market for high-speed rails, for example, there are typically a small number of firms capable of providing a system, a buyer is either a country or a region and hence is also limited in number, and the good has externalities since it is not simply a physical product but includes the operation and management of the system: Countries contemplating the adoption of a high-speed rail would be concerned with the rail system adopted by their neighbors if future connection between their systems is anticipated.[3]

It is well recognized that games with adoption externalities possess multiple equilibria. In our model, this corresponds to the potential multiplicity of Nash equilibria (NE) in the subgame played by the buyers after the posting of the prices by the firms. In the set of equilibria, our analysis makes use of two NE that are extreme as follows: The A-maximal NE is one in which the set of buyers who choose

[2]The explanation offered for such a pricing strategy in the literature is typically based on a monopoly platform which faces two sides with asymmetric price elasticities. Caillaud and Jullien (2001) and Ambrus and Argenziano (2009) offer explanation for the pricing strategy under price competition in a symmetric two-sided market with global externalities. See Section 2.

[3]The Association of the European Rail Industry (UNIFE) estimates that the average annual market volume of rail supply amounts to 150 billion Euro in 2011-2013 (World Rail Market Study). About 40% of the volume is in the form of management and maintenance of the system. The largest market is in Asia, where India alone plans highspeed rail systems in five different routes. Government-backed firms from Japan and China also compete fiercely for multiple rail routes in South-East Asia.

A is maximal in the sense that buyer i chooses A in it as long as he chooses A in *some* NE. The B-maximal equilibrium is one in which the set of buyers who choose B is maximal in the same sense. Our analysis builds on the assumption that the buyers play the B-maximal equilibrium after any deviation by firm A, and the A-maximal equilibrium after any deviation by firm B. We show that these behavioral patterns support the broadest spectrum of equilibria of the firms' price competition game by minimizing the profitability of deviations.

The paper is organized as follows: After discussing the related literature in Section 2, we formulate a model of price competition in Section 3. A leading example is given in Section 4. Section 5 considers the subgame played by the buyers in stage 2. Section 6 derives necessary conditions for an equilibrium in terms of the firms' payoffs. Section 7 provides a characterization of an equilibrium under linear externalities and introduces bipartition pricing strategies. The possibility of uniform pricing and marginal cost pricing under non-linear externalities is discussed in Section 8. Section 9 discusses the robustness of bipartition pricing strategies. Application of the analysis to two-sided markets is discussed in Section 10. Section 11 concludes with a discussion. All the proofs are collected in the Appendix.[4]

2 Related Literature

This paper contributes to two strands of literature. First, it contributes to the literature on network competition and two-sided markets through the introduction of local network externalities. Beginning with Katz and Shapiro (1985), most work on the topic supposes that the externalities are global in the sense that the adoption decision of any single buyer affects all other buyers equally.[5] In the context of two-sided markets, this implies that the participation of any agent on one side of the market equally affects the utility of all participating agents on the other side of the market.[6] In contrast, we suppose that the adoption decision of any buyer affects only his neighbors on the network. In two-sided markets, our formulation implies that the participation of any agent may have different effects on different agents on the other side of the market.[7]

Second, it presents a general analysis of price competition between suppliers of goods with local network externalities. Models of price competition under local network externalities include Banerji and Dutta (2009), Bloch and Quérou (2013), Blume *et al.* (2009) and Jullien (2011). Blume *et al.* (2009) and Bloch and Quérou

[4]While the main body of the paper focuses on equilibrium with market monopolization by one of the firms, an equilibrium with market segmentation is discussed in the supplementary material.

[5]See Gabszewicz and Wauthy (2004), Hagiu (2006), Ambrus and Argenziano (2009), Blume *et al.* (2009), Fjeldstad *et al.* (2010), Cabral (2011).

[6]See Armstrong (1998), and Laffont et al. (1998a,b).

[7]Ambrus and Argenziano (2009) analyze a market with global but asymmetric externalities in which each agent may have a different utility function over the size of participation in the same platform.

(2013) study price competition under local network externalities when market segmentation among the firms is exogenously given. Banerji and Dutta (2009) study price competition using a graph representation of local externalities when there is no price discrimination. A model of Stackelberg price competition by Jullien (2011) is most closely related to the present model and entails a very general specification of local externalities. Our specification of local externalities is more restrictive than that in Jullien (2011), but allows us to derive an explicit characterization of an equilibrium.[8]

The multiplicity of equilibria is often a central concern in games with network externalities. Since the pioneering work of Dybvig and Spatt (1983), this concern has led the literature to focus on such issues as implementing efficient or revenue maximizing equilibria under complete and incomplete information, intertemporal patterns of adoption decisions, as well as the validity of introductory pricing.[9] As mentioned in the Introduction, we abstract from the issue by supposing that whenever there is a deviating firm, the buyers coordinate on its least favorable NE. The literature makes different assumptions in this regard. For example, Ambrus and Argenziano (2009) assume that the agents' actions satisfy correlated rationalizability, which implies that they coordinate on the pareto-efficient alternative whenever there is one, and Jullien (2011, Assumption 2) assumes that a change in price offer by one firm to buyers outside its market segment does not affect the decisions of those inside it.

One key idea used in the present paper is that of *divide-and-conquer,* which has been studied by Segal (2003), Winter (2004) and Bernstein and Winter (2012) among others in contracting problems in which a single principal offers a contract to the set of agents whose participation decisions create externalities to other agents.[10]

3 Model

Two firms A and B compete for the set $I = \{1, \dots, N\}$ of $N \geq 3$ buyers. Adoption of either firm's good generates externalities to the buyers according to a buyer network. Formally, a buyer network is represented by a simple undirected graph G whose nodes correspond to the buyers, and adoption externalities exist between buyers i and j if they are *adjacent* in the sense that there is a link between i and j. When buyer j is adjacent to buyer i, we also say that j is i's *neighbor.*

[8]Sundararajan (2003), Candogan *et al.* (2012) and Bloch and Quérou (2013) each study monopoly pricing under local externalities.

[9]See Cabral et al. (1999), Park (2004), Sekiguchi (2009), Ochs and Park (2010), Aoyagi (2013), Parakhonyak and Vikander (2013), among others. Rohlfs (1974) provides a very early treatment of network externalities.

[10]A similar idea can be found in the study of an optimal marketing strategy under externalities in Hartline *et al.* (2008). A marketing strategy determines the order in which the monopolist approaches the set of buyers with private valuations as well as a sequence of contingent prices offered to them. See also Aoyagi (2010).

The buyer network G is *connected* in the sense that for any pair of buyers i and j, there exists a path from i to j. That is, there exist buyers $i_1, i_2, \ldots, i_m$, such that i_1 is adjacent to i, i_2 is adjacent to i_1, ..., and i_m is adjacent to j. For any buyer i in network G, denote by N_i the set of i's neighbors in G. The *degree* $d_i = |N_i|$ of buyer i is the number of i's neighbors. Define also M to be the number of links in G. Since each link counts twice when aggregating the number of degrees in G, we have $M = \frac{1}{2}\sum_{i\in I} d_i$.

For $r = 2, \ldots, N-1$, the network G is *r-regular* if all buyers have the same degree r, and regular if it is r-regular for some r. G is *cyclic* if it forms a single cycle, and *complete* if every pair of buyers are adjacent to each other.

The value of either firm's good to any buyer i is determined by the number of neighbors of i who adopt the same good. We denote by $v^n \geq 0$ the value of either good to any buyer when n of his neighbors adopt the same good. In particular, v^0 denotes the *stand-alone value,* or the value to any buyer of either good when none of his neighbors adopts the same good. Implicit in this assumption is that the two goods A and B are incompatible with each other since the value of either good to any buyer is assumed the same whether his neighbor adopts the other good or nothing. The value does not depend on the identity of a buyer or the identity of the firm who supplies the good. The externalities are positive in the sense that v^n is increasing in n. Let $\bar{d}$ be the highest degree in the network G: $\bar{d} = \max_{i\in I} d_i$. We will refer to the vector $(v^0, \ldots, v^{\bar{d}})$ as *externalities* and denote it by v. Denote by V_G the set of relevant externalities.

$$V_G = \left\{ v = (v^0, \ldots, v^{\bar{d}}) : c \leq v^0 \leq \cdots \leq v^{\bar{d}} \right\}.$$

Each firm supplies the good at the constant marginal cost $c \geq 0$ and no fixed cost. We will assume throughout that $c \leq v^0$ so that the firms can serve even a single buyer without making a loss.[11] The firms can perfectly price discriminate the buyers, and we let p_i and q_i denote the prices offered to buyer i by firm A and firm B, respectively. The price vectors $p = (p_i)_{i\in I} \in \boldsymbol{R}^N$ and $q = (q_i)_{i\in I} \in \boldsymbol{R}^N$ are quoted simultaneously and publicly observed. The buyers then simultaneously decide whether to buy either good, or buy neither.

Buyer i's action x_i is an element of the set $S_i = \{A, B, \emptyset\}$, where $\emptyset$ represents no purchase. Each firm's strategy is an element of $\boldsymbol{R}^N$, whereas buyer i's strategy σ_i is a mapping from the set $\boldsymbol{R}^{2N}$ of price vectors (p, q) to S_i. Given the price profile (p, q), buyer i's payoff under the action profile x is given by

$$u_i(x, p, q) = \begin{cases} v^{|\{j\in N_i : x_j = A\}|} - p_i & \text{if } x_i = A, \\ v^{|\{j\in N_i : x_j = B\}|} - q_i & \text{if } x_i = B, \\ 0 & \text{if } x_i = \emptyset, \end{cases} \tag{1}$$

[11] See Footnote 23 for one implication of the violation of this assumption.

Figure 1. Firm A's price offer p makes the choice of A (1) dominant for buyers 1 and 3 before it does so for buyer 2 (left), and (2) dominant for buyer 2 before it does so for buyers 1 and 3.

If we denote by $\sigma = (\sigma_i)_{i\in I}$ the buyers' strategy profile, the payoffs $\pi_A(p, q, \sigma)$ and $\pi_B(p, q, \sigma)$ of firms A and B, respectively, under the strategy profile (p, q, σ) are given by

$$\pi_A(p, q, \sigma) = \sum_{\{i:\, \sigma_i(p,q)=A\}} (p_i - c), \qquad \pi_B(p, q, \sigma) = \sum_{\{i:\, \sigma_i(p,q)=B\}} (q_i - c),$$

and buyer i's payoff $\pi_i(p, q, \sigma)$ under the strategy profile (p, q, σ) is given by $\pi_i(p, q, \sigma) = u_i(\sigma(p, q), p, q)$.

A price vector (p^*, q^*) and a strategy profile $\sigma = (\sigma_i)_{i\in I}$ together constitute a *subgame perfect equilibrium* (SPE) if given any price vector $(p, q) \in \boldsymbol{R}^{2N}$, the action vector $(\sigma_i(p, q))_{i\in I}$ is a Nash equilibrium of the subgame following (p, q), and given σ, the price vectors p^* and q^* are optimal against each other.

4 Leading Example

Consider the line network with three buyers 1, 2 and 3 in Figure 1.

• **Is marginal cost pricing consistent with an equilibrium?** Suppose that both firms engage in marginal cost pricing: $p^c = q^c = (c, c, c)$. If all buyers choose B, hence, the payoffs to buyers 1, 2 and 3 equal $v^1 - c$, $v^2 - c$ and $v^1 - c$, respectively. Suppose now that firm A offers the price vector $p = (p_1, p_2, p_3)$ such that

$$\begin{aligned} &v^0 - p_1 > \max\left\{v^1 - c, 0\right\}, \quad v^2 - p_2 > \max\left\{v^0 - c, 0\right\}, \\ &v^0 - p_3 > \max\left\{v^1 - c, 0\right\}. \end{aligned} \tag{2}$$

For buyer 1, choosing A is strictly dominant since it yields at least $v^0 - p_1$ whereas choosing $\emptyset$ yields 0 and choosing B yields $v^1 - c$ if his neighbor, buyer 2, chooses B. The same reasoning applies to buyer 3. It follows that $x_i = A$ is the unique outcome that survives the elimination of strictly dominated strategies for $i = 1$ and 3. Given this, however, buyer 2 finds A optimal since it yields $v^2 - p_2$, whereas $\emptyset$ yields 0 and B yields $v^0 - c$. In other words, $x_2 = A$ is the unique outcome that survives two rounds of elimination of strictly dominated strategies. If we give the orientation $i \to j$ to the link between buyers i and j when i precedes j in this iterative elimination process, then p satisfying (2) is depicted as in the left panel of

Figure 1. Note that such a p induces all buyers to choose A as long as they choose a rational response to the price offers. Hence, marginal cost pricing by both firms can only be consistent with an equilibrium if no such deviation by firm A yields a strictly positive payoff. The inequalities in (2) imply $\sum_i (p_i - c) < v^2 + v^0 - 2v^1$, but firm A can bring its payoff arbitrarily close to $v^2 + v^0 - 2v^1$ by offering p that barely satisfies (2). In other words, there exists a profitable deviation p satisfying (2) if

$$v^2 - v^0 + 2(v^0 - v^1) = v^2 + v^0 - 2v^1 > 0. \quad (3)$$

Suppose next that firm A offers the price vector $p = (p_1, p_2, p_3)$ such that

$$\begin{aligned} &v^1 - p_1 > \max\left\{v^0 - c, 0\right\}, \quad v^0 - p_2 > \max\left\{v^2 - c, 0\right\}, \\ &v^1 - p_3 > \max\left\{v^0 - c, 0\right\}. \end{aligned} \quad (4)$$

This time, buyer 2 finds $x_2 = A$ dominant in the first round of the iterative elimination process, and buyers 1 and 3 find $x_i = A$ dominant in the second round of the process. The orientation of the links hence is now given as in the right panel of Figure 1. The inequalities in (4) imply $\sum_i (p_i - c) < 2v^1 - v^2 - v^0$, and by the same logic as above, there exists a profitable deviation p satisfying (4) if

$$v^0 - v^2 + 2(v^1 - v^0) = 2v^1 - v^2 - v^0 > 0. \quad (5)$$

From (3) and (5), we see that marginal cost pricing is *not* an equilibrium if

$$2v^1 - v^2 - v^0 \neq 0, \quad (6)$$

which is true for a generic specification of (v^0, v^1, v^2). On the other hand, if the externalities are linear in the sense that $v^d - v^0 = hd$ for some $h > 0$ for every $d = 1, 2, \ldots$, then equality holds in (6). We can verify that under linear externalities, marginal cost pricing is indeed consistent with an equilibrium. In this network, hence, an equilibrium involving marginal cost pricing under linearity is not robust against slight perturbations in externalities. We show that this observation on marginal cost pricing extends to any network that is not a cycle or complete.

• **Pricing strategies consistent with a monopolization equilibrium?** Suppose that the externalities are either linear or satisfy (3), and consider the price pair (p^*, q^*) such that

$$p_1^* = q_1^* = v^0 - v^1 + c, \quad p_2^* = q_2^* = v^2 - v^0 + c, \quad \text{and} \quad p_3^* = q_3^* = v^0 - v^1 + c. \quad (7)$$

In other words, buyers 1 and 3 are offered markdowns from marginal cost c whereas buyer 2 is charged a markup from c. Note that when the externalities are linear, the price vectors in (7) reduce to $p = q = (-h + c, 2h + c, -h + c)$ so that the size of the markup or markdown equals the number of their neighbors who receive the opposite treatment from the firms.[12] Suppose that the buyers choose firm B under this price pair $\sigma(p^*, q^*) = (B, B, B)$, and that when firm A deviates from p^* to p, buyers choose A only when they find A iteratively strictly dominant under (p, q^*). By (3), firm B's payoff is positive under (p^*, q^*). Consider firm A's price vector p that makes A dominant for buyer 1 in the first round, then iteratively dominant for buyers 3 and 2 in this order in subsequent rounds. p then must satisfy

$$v^0 - p_1 > \max\left\{v^1 - q_1^*, 0\right\}, \quad v^2 - p_2 > \max\left\{v^0 - q_2^*, 0\right\},$$
$$v^0 - p_3 > \max\left\{v^0 - q_3^*, 0\right\}.^{13}$$

Combining these inequalities together, we have

$$\sum_i (p_i - c) < \left(v^0 + v^2 - 2v^1\right) + \sum_i (q_i^* - c) + (v^1 - q_2^*).$$

When the externalities are linear as defined above, the first two terms on the right-hand side equal zero whereas the third term is < 0 as can be readily verified. It follows that no such deviation p is profitable also when the externalities are approximately linear. If, on the other hand, firm A's price vector p makes A dominant for buyers 2 in the first round and iteratively dominant for buyers 1 and 3 a subsequent round as in the right panel of Figure 1, p satisfies

$$v^1 - p_1 > \max\left\{v^0 - q_1^*, 0\right\}, \quad v^0 - p_2 > \max\left\{v^2 - q_2^*, 0\right\},$$
$$v^1 - p_3 > \max\left\{v^0 - q_3^*, 0\right\}.$$

These inequalities together imply that

$$\sum_i (p_i - c) < -\left(v^0 + v^2 - 2v^1\right) + \sum_i (q_i^* - c) = 0.$$

Hence, no such p is profitable either. We can indeed verify that firm A has no profitable deviation, and conclude that the pricing strategies in (7) are consistent with a monopolization equilibrium when the externalities are linear, or approximately linear and also satisfy (3). Put differently, the equilibrium price vector (p, q) under linearity in (7) is robust against slight perturbations in externalities which satisfy (3). This marks a sharp contrast with the marginal cost pricing equilibrium discussed above.[14]

[12] This is a special case of the bipartition equilibrium introduced in Section 7.

[13] When buyer 1 finds A dominant in round 1, buyer 2 finds B dominated by $\emptyset$ in round 2 since he has only one neighbor (*i.e.*, buyer 3) who may choose B and hence $v^1 - q_2^* < 0$. This further implies that buyer 3's payoff from choosing B in any subsequent round of the iteration process equals $v^0 - q_3^*$ instead of $v^1 - q_3^*$, as indicated in the third inequality. With $x_2 = B$ not rationalizable for buyer 2, firm A can make $x_3 = A$ iteratively dominant for buyer 3 more easily (*i.e.*, at a higher price p_3) than in the hypothetical scenario where $x_2 = B$ is rationalizable for buyer 2. However, it can be shown generally that no deviation p by firm A is profitable if it eliminates B before A becomes dominant. See the discussion after Proposition 7.2 in Section 7.

[14] We can also verify that there exists no equilibrium in which the two firms segment the market. The supplementary material provides a necessary condition for the existence of such an equilibrium. This condition fails in the network of this example.

5 Nash Equilibrium in the Buyers' Game

In this section, we fix the price vector (p, q), and consider an equilibrium of the buyers' subgame following (p, q) in which the set of actions of each buyer i equals $S_i = \{A, B, \emptyset\}$, and his payoff function u_i is defined by (1). The simultaneous-move game $(I, S = \prod_{i \in I} S_i, (u_i)_{i \in I})$ among the buyers is one of strategic complementarities given that any buyer's incentive to choose A (resp. B) increases when more buyers choose A (resp. B). As such, the buyers' game typically has multiple equilibria. Among them, we are interested in two pure Nash equilibria x^A and x^B that are extreme in the set of buyers who choose either A or B. Formally, x^A is A-*maximal* in the sense that if y is any Nash equilibrium and if any buyer i chooses A in y (*i.e.*, $y_i = A$), then he also chooses A in x^A ($x_i^A = A$). Likewise, x^B is B-*maximal* in the sense that if y is any Nash equilibrium and if any buyer i chooses B in y ($y_i = B$), then he also chooses B in x^B ($x_i^B = B$).[15, 16]

Formally, if we define D_A (resp. D_B and $D_\emptyset$) to be the set of buyers for whom $x_i = A$ (resp. $x_i = B$ and $x_i = \emptyset$) is iteratively dominant, and $D \equiv D_A \cup D_B \cup D_\emptyset$, then any buyer $i \in D$ must choose his iteratively dominant action in any NE. It follows that any pair of NE may be different from each other only in the actions chosen by buyers not in D. The A-maximal and B-maximal NE can then be constructed by simply having all buyers for whom A and B are rationalizable choose A and B, respectively, as seen in the following proposition.

Proposition 5.1 *(Maximal NE) Define x^A and x^B by*

$$x_i^A = \begin{cases} A & \text{if } x_i = A \text{ is rationalizable,} \\ B & \text{if } x_i = B \text{ is iteratively dominant,} \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$x_i^B = \begin{cases} A & \text{if } x_i = A \text{ is iteratively dominant,} \\ B & \text{if } x_i = B \text{ is rationalizable,} \\ \emptyset & \text{otherwise,} \end{cases}$$

Then x^A and x^B are the A-maximal and B-maximal NE, respectively.

We denote by σ^A the buyers' strategy profile that chooses the A-maximal NE for any price pair (p, q), and by σ^B the strategy profile that chooses the B-maximal NE after any (p, q). It is clear from Proposition 5.1 that when the buyers play σ^B,

[15] Formally, the game is supermodular when the set S_i of actions of each buyer is endowed with the ordering $A \succ \emptyset \succ B$. The set of NE in a supermodular game has maximal and minimal elements with respect to the partial ordering $\succ_S$ on S induced by $\succ$. See Topkis (1998).

[16] Any NE survives the iterative elimination of strictly dominated actions, and in a finite supermodular game, any strategy profile x that survives this process lies between x^A and x^B: $x^A \succcurlyeq_S x \succcurlyeq_S x^B$. See Milgrom and Roberts (1990).

the set of buyers who choose A under any (p, q) equals $D_A(p, q)$. Likewise, when they play σ^A, the set of buyers who choose B equals $D_B(p, q)$:

$$\left\{i : \sigma_i^B(p, q) = A\right\} = D_A(p, q), \quad \text{and} \quad \left\{i : \sigma_i^A(p, q) = B\right\} = D_B(p, q).$$

Suppose now that the price vector (p^*, q^*) is given, and that we want to examine if any deviation by firm A to an alternative price vector p is profitable. A different deviation p will induce a different iterative elimination process, and as will be seen, the exact order in which the buyers find the choice of A iteratively dominant under the price vector (p, q^*) determines whether this deviation by firm A is profitable or not. For this reason, our analysis requires the description of the iterative elimination of dominated strategies in the buyers' game in some more detail. For any product subset $S' = \prod_i S'_i \subset S$ of action profiles such that $S'_i \neq \emptyset$, buyer i's action $x_i \in S'_i$ is (strictly) *dominated* in S' (by another pure action) if there exists $x'_i \in S'_i$ such that

$$u_i(x_i, x_{-i}) < u_i(x'_i, x_{-i}) \text{ for every } x_{-i} \in S'_{-i}.$$

$x_i \in S'_i$ is *dominant* in S' if any other action $x'_i \in S'_i$ is dominated in S' (by x_i). Let $S^0 = S$, and for $k = 1, 2, \ldots$, let S_i^k be the set of buyer i's *k-rationalizable* actions, *i.e.,* the actions that survive the elimination of *all* dominated strategies in S^{k-1}. If x_i is the unique k-rationalizable action for buyer i for the first time in the iteration process (*i.e.,* $S_i^{k-1} \supsetneq S_i^k = \{x_i\}$), we say that action x_i is *k-dominant.* Since each buyer has at most two dominated actions, the above process stops in or before $2N$ rounds. Let then K be the last round of the iteration process: $S^{K+1} = S^K$. Define

$$\begin{aligned} D_A^k &= \{i \in I : x_i = A \text{ is } k\text{-dominant}\}, \quad \text{and} \\ R_A^k &= \{i \in I : x_i = A \text{ is } k\text{-rationalizable}\}. \end{aligned} \tag{8}$$

For any $k \geq 1$, $\cup_{\ell=1}^k D_A^\ell$ is the set of buyers for whom $x_i = A$ is the unique k-rationalizable action, and the set of buyers for whom $x_i = A$ is *iteratively dominant* equals $D_A \equiv \cup_{k=1}^K D_A^k$. On the other hand, R_A^K is the set of buyers for whom $x_i = A$ is *rationalizable.* D_B, $D_\emptyset$, R_B^K and $R_\emptyset^K$ have similar interpretations. Since the above iteration process depends on the price profile (p, q), we often write $D_A^k(p, q)$ and so on to make this dependence explicit.

For any i, define $\alpha_i^1 = 0$ and $\beta_i^1 = d_i$. For $k \geq 2$, define α_i^k to be the number of i's neighbors j for whom $x_j = A$ is the unique $(k-1)$-rationalizable action, and β_i^k to be the number of his neighbors j for whom $x_j = B$ is $(k-1)$-rationalizable. $x_i = A$ is the unique k-rationalizable action for buyer i if and only if the payoff from choosing A along with α_i^k of his neighbors is higher than that from choosing B along with β_i^k of his neighbors, or from choosing $\emptyset$. In other words, $i \in \cup_{\ell=1}^k D_A^\ell$ if and only if

$$v^{\alpha_i^k} - p_i > \max\left\{v^{\beta_i^k} - q_i, 0\right\},$$

or equivalently,

$$p_i < \min\left\{v^{\alpha_i^k} - v^{\beta_i^k} + q_i, v^{\alpha_i^k}\right\}. \tag{9}$$

If $x_i = A$ is k-dominant ($i \in D_A^k$) for buyer i and $x_j = A$ is either ℓ-dominant ($j \in D_A^\ell$) for $\ell > k$ or not iteratively dominant ($j \notin D_A$) for buyer j, we say that i *precedes* j in the elimination process, and write $i \prec_{(p,q)} j$ or simply $i \prec j$.[17] When buyers i and j are adjacent, the precedence relation $\prec$ induces an orientation of the link between i and j: The link ij is given the orientation $i \to j$ if $i \prec j$.

6 Subgame Perfect Equilibrium

We now turn to the original two-stage game including the firms. We first observe that if a price vector (p^*, q^*) is sustained in some SPE, then it must be sustained in an SPE in which the buyers choose an extreme response to either firm's deviation. Formally, a strategy profile σ of the buyers is *extremal with respect to* (p^*, q^*) if

$$\sigma(p,q) = \begin{cases} \sigma^B(p,q) & \text{if } p \neq p^* \text{ and } q = q^*, \\ \sigma^A(p,q) & \text{if } p = p^* \text{ and } q \neq q^*. \end{cases} \tag{10}$$

In other words, when firm A unilaterally deviates from p^*, then all buyers play the B-maximal NE that least favors firm A, and when firm B unilaterally deviates from q^*, then they play the A-maximal NE that least favors firm B.

Proposition 6.1 *(Bang-bang property) (p^*, q^*) is an SPE price vector if and only if (p^*, q^*, σ) is an SPE for σ that is extremal with respect to (p^*, q^*).*

Consider next firm A's best response p to B's price q when the buyers play the B-maximal strategy σ^B. We say that the set J of buyers is *independent* if no pair of buyers in J are adjacent to each other.[18] The following lemma shows that if p is any "optimal" deviation by firm A, then $D_A^k(p,q)$ is independent. In other words, p is never optimal if there exists a pair of adjacent buyers who belong to the same round of the elimination process. Intuitively, this is because making A dominant for adjacent buyers i and j simultaneously requires offering lower prices to both of them than making $x_i = A$ dominant for buyer i first, then making $x_j = A$ dominant for buyer j next conditional on i choosing $x_i = A$. For illustration, consider the line network of Section 4. If firm A offers p that makes A dominant for buyers 1 and 2 simultaneously, then p must satisfy

$$p_1 < \min\{v^0 - v^1 + q_1^*, v^0\} \text{ and } p_2 < \min\{v^0 - v^2 + q_2^*, v^0\}.$$

On the other hand, if it offers p' that makes A dominant for buyer 1 first and then buyer 2 next, p' should satisfy

$$p_1' < \min\{v^0 - v^1 + q_1^*, v^0\} \text{ and } p_2' < \min\{q_2^*, v^1\}.$$

It is clear that p' can be taken so that $p_1 + p_2 < p_1' + p_2'$.

[17] For simplicity, we define the precedence relationship only for A.
[18] J is also independent if it is a singleton.

Lemma 6.2 *(Strict precedence between adjacent buyers) Let the price vector* (p, q) *be given and suppose that buyers* i *and* j *are adjacent. If* i *and* j *both find* $x_i = x_j = A$ k*-dominant for the same* k *under* (p, q)*, then there is another price vector* p' *such that firm* A*'s payoff under* (p', q) *is strictly higher than that under* (p, q)*, and that* i *precedes* j *under* (p', q)*:* $\pi_A(p', q, \sigma^B) > \pi_A(p, q, \sigma^B)$ *and* $i \prec_{(p',q)} j$*.*

Note that p may or may not entail strict precedence between non-adjacent buyers. Note in general that firm A's optimal deviation may be to make the choice of A iteratively dominant for only a subset of buyers, and that the buyers may choose A even if it is not iteratively dominant. In this sense, if we consider p that makes A iteratively dominant for every buyer, then it gives a lower bound for firm A's deviation payoff. Formally, given the price vector q of firm B, p is firm A's *divide-and-conquer* (DC) price vector against q if it (1) entails strict precedence between every pair of adjacent buyers ("divide"), and (2) makes A iteratively dominant for all buyers ("conquer"). If p is a DC price vector, hence, the entire buyer network G is given an orientation $\rightarrow$ induced by the precedence relation $\prec$.

Let a price vector (p, q) be given. For any buyer $i \in D_A(p, q)$ and $\prec \equiv \prec_{(p,q)}$, define

$$s_i^{\prec} = |\{j \in N_i : j \prec i\}| \tag{11}$$

to be the number of i's neighbors who precede him in the elimination process. When no confusion arises, we omit the dependence of $s_i^{\prec}$ on $\prec$ and simply write s_i. We refer to s_i as buyer i's *in-degree* and $d_i - s_i$ as i's *out-degree.*

In terms of the orientation $\rightarrow$ induced by $\prec$, buyer i's in-degree is simply the number of links directed toward i, and his out-degree is the number of links directed toward i's neighbors. Recall also that α_i^k is the number of i's neighbors for whom $x_i = A$ is the unique $(k-1)$-rationalizable action (*i.e.,* $\{A\} = S_i^{k-1}$), and β_i^k is the number of his neighbors for whom $x_i = B$ is $(k-1)$-rationalizable (*i.e.,* $B \in S_i^{k-1}$). Hence, if $x_i = A$ is k-dominant for buyer i (*i.e.,* $i \in D_A^k$), then

$$s_i = \alpha_i^k \quad \text{and} \quad d_i - s_i \geq \beta_i^k.\text{[19]} \tag{12}$$

In the example of Section 4, the sequence of in-degrees is given by $(s_1, s_2, s_3) = (0, 2, 0)$ in the left panel of Figure 1 and $(s_1, s_2, s_3) = (1, 0, 1)$ in the right panel. When $\prec$ induces an orientation to every link of the network G, then the sum of in-degrees of all buyers and the sum of out-degrees of all buyers both equal the total number M of links in G:

$$\sum_{i \in I} s_i = \sum_{i \in I} (d_i - s_i) = M. \tag{13}$$

Define now O_G to be the set of all possible orientations $\rightarrow$ of the buyer network G. If $s = (s_i^{\prec})_{i \in I}$ is the sequence of in-degrees for the relation $\prec$, then the corresponding

[19] Note that $d_i - s_i > \beta_i^k$ if i has a neighbor j for whom $x_j = B$ is not $(k-1)$-rationalizable but $x_j = A$ is not the unique $(k-1)$-rationalizable action, *i.e.,* if $S_j^{k-1} = \{A, \emptyset\}$ or $\{\emptyset\}$.

sequence of out-degrees $d - s \equiv (d_i - s_i^{\prec})_{i \in I}$ equals the sequence of in-degrees for the reverse relation $\prec'$ such that $i \prec' j$ if and only if $j \prec i$.

By (9) and (12), a sufficient condition for the dominance of $x_i = A$ for i in (9) is given by

$$\begin{aligned} p_i &< \min \left\{ v^{s_i} - v^{d_i - s_i} + q_i,\, v^{s_i} \right\} \\ &= v^{s_i} - v^{d_i - s_i} + \min \left\{ q_i,\, v^{d_i - s_i} \right\}. \end{aligned} \tag{14}$$

By offering p that barely satisfies (14) for every i, hence, firm A captures all the buyers ($D_A(p,q) = I$) and achieves the profits arbitrarily close to

$$\sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} + \min \left\{ q_i,\, v^{d_i - s_i} \right\} - c \right). \tag{15}$$

We hence have the following lemma that gives a lower bound for each firm's equilibrium payoff.

Lemma 6.3 *(Lower bound on equilibrium payoffs) If (p^*, q^*, σ) is an SPE, then*

$$\begin{aligned} \pi_A(p^*, q^*, \sigma) &\geq \max_{\prec \in O_G} \sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} + \min \left\{ q_i^*,\, v^{d_i - s_i} \right\} - c \right), \\ \pi_B(p^*, q^*, \sigma) &\geq \max_{\prec \in O_G} \sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} + \min \left\{ p_i^*,\, v^{d_i - s_i} \right\} - c \right). \end{aligned} \tag{16}$$

Lemma 6.3 shows that a firm's payoff from DC is closely linked to the value of

$$\sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} \right). \tag{17}$$

(17) is the key quantity referred to as firm A's *benchmark payoff given* $\prec$. Since $c \leq v^0 \leq v^{d_i - s_i}$, (15) reduces to (17) when firm B engages in marginal cost pricing $q^c = (c, \ldots, c)$. The benchmark payoff can hence be interpreted as the maximal payoff achieved by employing DC against marginal cost pricing.[20]

7 Linear Externalities

When there are no adoption externalities $c \leq v^0 = \cdots = v^{\bar{d}}$, it is clear that a subgame perfect equilibrium price (p^*, q^*) is unique and equal to the marginal cost: $p^* = q^* = (c, \ldots, c)$. In this section, we will examine if and how this result can be

[20] Note that the benchmark payoff is determined by the relation $\prec$ alone and independent of the particular price vector p.

extended when the externalities increase linearly in the size of adoption. Specifically, we say that the externalities $v = (v^0, \ldots, v^{\bar{d}})$ are *linear* if there exists $h > 0$ such that

$$v^d - v^0 = dh \quad \text{for every } d = 0, 1, \ldots, \bar{d}.$$

Linearity is a working assumption in many models of network externalities in the literature.[21] We first establish that marginal cost pricing survives as an equilibrium under linear externalities.

Proposition 7.1 *(MC pricing under linear externalities) Let G be an arbitrary buyer network. If the externalities v are h-linear for some $h > 0$, $p^* = q^* = (c, \ldots, c)$ is an equilibrium price vector.*

To see the intuition behind Proposition 7.1, suppose that firm B monopolizes the market under $p^* = q^* = (c, \ldots, c)$ and that the B-extremal NE σ^B is played when firm A deviates. If firm A offers a DC price vector p, then by (15), its payoff satisfies

$$\begin{aligned}\sum_{i \in I}(p_i - c) &< \sum_{i \in I}\left(v^{s_i} - v^{d_i - s_i} + \min\left\{c,\, v^{d_i - s_i}\right\} - c\right) \\ &= \sum_{i \in I}\left(v^{s_i} - v^{d_i - s_i}\right) && \Leftarrow c \le v^0 \\ &= 0. && \Leftarrow \text{linearity and (13)}\end{aligned}$$

It can also be checked that firm A cannot profitably deviate by offering p that attracts only a subset of buyers.

Unlike in the case of no externalities, however, there exists a large multiplicity of equilibria in the price competition game with linear externalities. The following proposition provides a complete characterization of the set of pricing strategies consistent with a monopolization equilibrium when the two firms offer the same price vector. For any $J \subset I$, denote by $L(J)$ the number of links in J, and for any J, $J' \subset I$ with $J \cap J' = \emptyset$, denote by $L(J, J')$ the number of links between J and J'.

Proposition 7.2 *Let G be an arbitrary buyer network and suppose that the externalities v are h-linear for some $h > 0$. Let $z = (z_i)_{i \in I}$ be any price vector.*

1) $(p^*, q^*) = (z, z)$ is consistent with a monopolization equilibrium if

$$\sum_{i \in J}(z_i - c) \le hL(J, I \setminus J) \text{ for every } J \subsetneq I \text{ and } \sum_{i \in I}(z_i - c) = 0. \tag{18}$$

2) Suppose that $v^0 > c$. If $(p^*, q^*) = (z, z)$ is consistent with a monopolization equilibrium, then z satisfies (18).

[21] See, for example, Caillaud and Jullien (2003), Ambrus and Argenziano (2009), Candogan *et al.* (2012) and Bloch and Quérou (2013).

One key step of the proof for Proposition 7.2(1) is Lemma A.2 in the Appendix which shows that no deviation p is profitable unless it satisfies the following condition: Under (p, q^*), every buyer i has either $x_i = A$ as the unique k-rationalizable action, or $x_i = B$ as one of the k-rationalizable actions. This point is illustrated in the line network of Section 4. Suppose that firm B monopolizes the market with the price vector q^* satisfying q_1^*, $q_3^* < 0$, $q_2^* > v^1 = v^0 + h$, and $\sum_i (q_i^* - c) = 0$. Suppose also that firm A deviates to p such that under (p, q^*), all three buyers find A iteratively dominant with the precedence relationship $1 \prec 3 \prec 2$ (as in the right panel of Figure 1). Since buyer 1 finds $x_1 = B$ dominated in round 1, and since $q_2^* > v^1$, buyer 2 finds $x_2 = B$ dominated in round 2 so that it is not 2-rationalizable. However, $x_2 = A$ is not 1- or 2-dominant and hence is not the unique 2-rationalizable action for buyer 2. This p hence violates the above condition. Furthermore, $\beta_3^2 = 0$ since buyer 3 has no neighbor for whom B is 2-rationalizable. It follows that p should satisfy:

$$
\begin{aligned}
p_1 &< \min\{v^0 - v^1 + q_1^*, v^0\} = v^0 - v^1 + q_1^*, \\
p_3 &< \min\{v^0 - v^0 + q_3^*, v^0\} = q_3^*, \\
p_2 &< \min\{v^2 - v^0 + q_2^*, v^2\} = v^2,
\end{aligned}
$$

where the last equality holds since $q_2^* > v^1 > v^0$. It hence follows that p is not a profitable deviation since $\sum_i (p_i - c) < v^0 - v^1 + v^2 + q_1^* + q_3^* - 3c = (v^0 + h) - q_1^* < 0$.

The above observation in turn implies $\alpha_i^k + \beta_i^k = d_i$ for every i and k since every neighbor of i either has A as his unique $(k-1)$-rationalizable action (who counts towards α_i^k) or B as one of the $(k-1)$-rationalizable actions (who counts towards β_i^k).[22] This substantially simplifies checking the profitability of deviations since the deviating firm's payoff can be evaluated in terms of the indegrees and the markups/markdowns imposed by firm B:

$$
\sum_{i \in D_A} (p_i - c) < \sum_{k=1}^{K} \sum_{i \in D_A^k} (v^{\alpha_i} - v^{d_i - \alpha_i^k} + q_i^* - c) = \sum_{i \in D_A} (v^{s_i} - v^{d_i - s_i} + q_i^* - c).
$$

It is readily seen (Lemma A.1 in the Appendix) that $\sum_{i \in D_A} (v^{s_i} - v^{d_i - s_i}) = -hL(D_A, I \setminus D_A)$ so that no deviation p is profitable against q^* which satisfies (18) for $J = D_A$.

The intuition behind Proposition 7.2(2) is rather simple: Suppose for example that $J = \{i\}$ and that $q_i^* > hL(J, I \setminus J) = hd_i$. In this case, $\min\{v^0 - v^{d_i} + q_i^*, v^0\} = v^0$ and hence firm A can profitably attract buyer i (*i.e.*, $i \in D_A(p, q^*)$) by offering p with $p_i \in (c, v^0)$ and $p_j = c$ for $j \neq i$. This observation basically extends to the case where $|J| \geq 2$.

Among the large set of pricing strategies consistent with a monopolization equilibrium identified in Proposition 7.2, we are interested in a particular class that

[22] Note that this equality fails in the example given since $\alpha_3^2 + \beta_3^2 = 0 < d_3$.

entails price discrimination of buyers as follows. Specifically, consider price discrimination based on a binary partition of the buyer set: Buyers in one subset are charged markups and those in the other subset are offered discounts. Formally, a *bipartition* of the set I of buyers is an ordered pair (I_1, I_2) of subsets I_1, $I_2 \subset I$ such that $I_2 = I \setminus I_1$. For any $i \in I$, denote by d_i^k the number of his neighbors in set I_k:

$$d_i^1 = |N_i \cap I_1|, \quad \text{and} \quad d_i^2 = |N_i \cap I_2|.$$

We say that $z = (z_i)_{i \in I}$ is a *bipartition price vector* given the bipartition (I_1, I_2) if

$$z_i = \begin{cases} d_i^2 h + c & \text{if } i \in I_1, \\ -d_i^1 h + c & \text{if } i \in I_2. \end{cases} \tag{19}$$

Hence, every buyer in I_1 is charged a markup and every buyer in I_2 is offered a markdown from c. Furthermore, the size of the markup or markdown to each buyer is proportional to the number of his neighbors in the other subset. We can also verify that the markups and markdowns sum up to zero: $\sum_{i \in I} (z_i - c) = 0$. A profile (p^*, q^*, σ) is a *bipartition equilibrium with monopolization given* (I_1, I_2) if it is an SPE with monopolization by one of the firms and $(p^*, q^*) = (z, z)$ for some bipartition price vector z given (I_1, I_2). Proposition 7.3 below establishes the existence of a bipartition equilibrium under linear externalities.

Proposition 7.3 *(Bipartition equilibrium) Let G be an arbitrary buyer network and suppose that the externalities v are h-linear for some $h > 0$. For any bipartition (I_1, I_2) of the set I, let z be the bipartition price vector given (I_1, I_2). Then $(p, q) = (z, z)$ is consistent with a bipartition equilibrium.*

Since the choice of the bipartition is arbitrary, the total number of bipartition equilibria is still very large. It is interesting to note that the bipartition equilibrium of Proposition 7.3 reduces to the marginal cost pricing equilibrium of Proposition 7.1 when the bipartition is given by $(I_1, I_2) = (\emptyset, I)$: In this case, no buyer has a neighbor in the other subset, and hence a bipartition equilibrium entails offering the marginal cost to every buyer.

8 Marginal Cost Pricing under Generic Externalities

We next consider the consequence of introducing some generic property of externalities. The externalities $v = (v^0, \ldots, v^{\bar{d}})$ are *generic* if for any orientation $\rightarrow \in O_G$ of the buyer network,

$$d - s \text{ is } \textit{not} \text{ a permutation of } s \quad \Rightarrow \quad \sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} \right) \neq 0. \tag{20}$$

Since the sequence $d - s$ of out-degrees for any orientation $\rightarrow$ equals the sequence s of in-degrees for the reverse orientation as noted before, (20) implies that the benchmark payoff for some orientation is strictly positive as long as there exists an orientation in O_G for which s and $d - s$ are not permutations of each other. The externalities v satisfying (20) are indeed generic in the set V_G of possible externalities in G. The condition (6) of the leading example in Section 4 corresponds to (20). The following lemma shows that under (20), the benchmark payoff is strictly positive for some orientation if and only if the buyer network is neither cyclic nor complete.

Lemma 8.1 *(Positivity of the benchmark payoff) If the buyer network G is neither cyclic nor complete, then s and $d - s$ are not permutations of each other for some orientation $\rightarrow$ so that*

$$\max_{\rightarrow \in O_G} \sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} \right) > 0.$$

Lemma 8.1(1) is illustrated in Figure 2 for a regular network. The sequences of in-degrees and out-degrees of buyers 1, 2 and 4 in the left panel equal

$$(s_1, s_2, s_4) = (0, 1, 1),\ (d_1 - s_1, d_2 - s_2, d_4 - s_4) = (4, 3, 3),$$

respectively, whereas those in the right panel equal

$$(s_1, s_2, s_4) = (0, 2, 0),\ (d_1 - s_1, d_2 - s_2, d_4 - s_4) = (4, 2, 4),$$

respectively. Since the sequences of in-degrees and out-degrees of all other buyers are the same in both panels, we see that if s and $d - s$ are permutations of each other in the left-panel, they cannot be so in the right-panel, and vice versa.

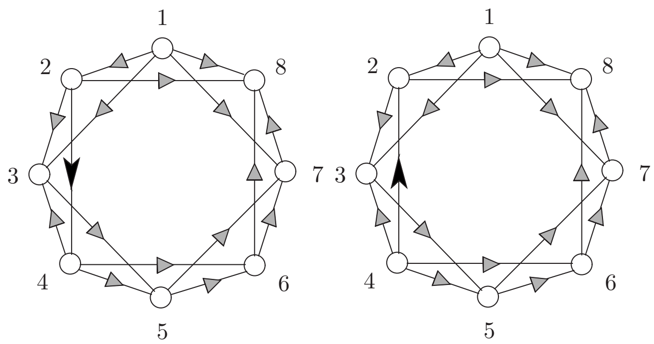

Figure 2. Illustration of Proposition 8.1(1) for a regular network

Suppose now that firm B monopolizes the market with uniform pricing q^* ($q_1^* = \cdots = q_N^* \geq c$). In this case, since $c \leq v^0 \leq v^{d_i - s_i}$ by assumption, the second term on the right-hand side of (15) is non-negative:

$$\sum_{i \in I} \left(\min \left\{ q_i,\ v^{d_i - s_i} \right\} - c \right) \geq 0. \tag{21}$$

Since $\pi_A(p^*, q^*, \sigma) = 0$, Lemma 8.1 and (21) together imply that (16) is violated for firm A, implying the impossibility of a monopolization equilibrium with uniform pricing.[23] As seen in the following proposition, further inspection of (21) shows that the monopolizing firm must offer a markdown from c to some buyer and also charge a markup above v^0 to another buyer. Furthermore, the largest externalities in a network cannot be too small compared with v^0. The last condition is a non-trivial restriction for networks in which the largest degree $\bar{d}$ is small as in line networks.

Proposition 8.2 *(No monopolization equilibrium with uniform pricing) Suppose that the buyer network G is neither complete nor cyclic, and that the externalities v are generic (20). Let (p^*, q^*, σ) be an equilibrium with monopolization by one firm, say firm B: $\sigma(p^*, q^*) = (B, \ldots, B)$. Then $\min_i q_i^* < c$, $\max_i q_i^* > v^0$ and $v^{\bar{d}} > 2v^0 - c$.*

As depicted in Figure 2, there are networks that are not cyclic or complete, but are symmetric with respect to every buyer. It follows from Proposition 8.2 that those identical buyers must also face price discrimination.

We are now ready to state our main result in this section. Suppose that both firms engage in marginal cost pricing $p^* = q^* = (c, \ldots, c)$ to all the buyers. By Proposition 8.2, it cannot be consistent with a monopolization equilibrium unless the network is complete or cyclic. On the other hand, it is indeed consistent with a monopolization equilibrium in these classes of networks essentially because for any precedence relation $\prec$, the associated sequences of indegrees and outdegrees are permutations of each other.

Proposition 8.3 *(MC pricing in cyclic and complete networks) Let a buyer network G be given and the externalities v are generic (20). $(p^*, q^*) = (z, z)$ for $z = (c, \ldots, c)$ is consistent with a monopolization equilibrium if and only if G is either cyclic or complete.*

9 Robustness of a Bipartition Equilibrium

We now examine the robustness of the equilibrium under linear externalities against slight perturbation in the externalities. To this end, we consider externalities that are approximately linear: For $h > 0$, the externalities $v = (v^0, \ldots, v^{\bar{d}})$ are *ε-close to h-linear* if

$$|v^d - v^0 - hd| < \varepsilon \text{ for } d = 1, \ldots, \bar{d}.$$

[23] When $c > v^0$, marginal cost pricing can be consistent with an equilibrium in some cases. In the line network of Section 4, for example, if $v^0 < c \leq \frac{1}{3}(2v^1 + v^2)$ and $c \geq \max\{v^2 - 2(v^1 - v^0), v^1 - \frac{1}{2}(v^2 - v^0)\}$, then there exists an equilibrium with $p^c = q^c = (c, c, c)$. Intuitively, this is so because (21) fails when $c > v^0$ so that a firm's payoff from a DC price vector may never exceed its benchmark payoff as defined in (17).

We also say that the (sequence of) externalities $(v(n))_{n\in\mathbf{N}}$ *approach h-linearity* if for any $\varepsilon > 0$, there exists n such that $v(n')$ is ε-close to h-linear for $n' \geq n$. Given $h > 0$, an equilibrium (p, q, σ) under h-linearity is *robust* if there exists a non-degenerate set V^* of externalities such that (i) V^* contains h-linear externalities, and (ii) for every $v \in V^*$, there exists an equilibrium (p^v, q^v, σ^v) such that $(p^v, q^v) \to (p, q)$ as v approaches h-linearity from within V^* and that $\sigma^v(p^v, q^v) = \sigma(p, q)$. The following result readily follows from Propositions 8.2 and 8.3.

Proposition 9.1 *(Non-robustness of MC pricing) Suppose that $v^0 > c$ and let $(p, q) = (z, z)$ for $z = (c, \ldots, c)$. Then the monopolization equilibrium (p, q, σ) under h-linearity is robust if and only if the buyer network G is a cycle or complete.*

In contrast with the non-robustness of MC pricing seen in Proposition 9.1, Proposition 9.2 below identifies a sufficient condition for bipartition pricing to be robust. Consider the set of precedence relations $\prec$ according to which every buyer in I_1 precedes every buyer in I_2:

$$i \prec j \text{ if } i \in I_1 \text{ and } j \in I_2,$$

and let O^*_G be the corresponding set of orientations of the buyer network. For each orientation $\to \in O^*_G$ and each degree $d = 1, \ldots, \bar{d}$, define $\lambda^{\to}_d$ to be "the number of buyers whose indegree equals d" minus "the number of buyers whose outdegree equals d":

$$\lambda^{\to}_d = |\{i : s_i = d\}| - |\{i : d_i - s_i = d\}| \quad \text{and} \quad \lambda^{\to} = (\lambda^{\to}_1, \ldots, \lambda^{\to}_{\bar{d}}). \tag{22}$$

Clearly, $\lambda^{\to} = 0$ if and only if s and $d - s$ are permutations of each other under the orientation $\to$. The sufficient condition is described in terms of these vectors $\lambda^{\to}$. A subset $J \subset I$ is *maximally independent* if (1) it is independent (*i.e.,* contains no pair of adjacent buyers), and (2) there exists no $J' \supsetneq J$ that is independent.

Proposition 9.2 *Let the network G be given, and (I_1, I_2) be the bipartition of the set of buyers such that I_1 is maximally independent. Suppose that $h > v^0 - c > 0$. The bipartition equilibrium (p, q, σ) under h-linearity is robust if no convex combination of the collection of vectors $\Lambda \equiv \{\lambda^{\to} : \lambda^{\to} \neq 0, \to \in O^*_G\}$ equals zero.*

The interpretation of Proposition 9.2 is as follows: Since every buyer in I_1 precedes every buyer in I_2 according to any orientation in O^*_G, if $\to \in O^*_G$, then its reverse is not an element of O^*_G. This is important since the non-robustness of the marginal cost pricing equilibrium (Lemma 8.1 and Proposition 8.3) results from the fact that the benchmark payoff under one orientation is always the negative of that under the reverse orientation. It may however be the case that even within O^*_G, there exists an orientation that generates the same sequence of indegrees (up to permutations) as that generated by the reverse of some $\to \in O^*_G$. The condition that no convex combination of vectors in $\{\lambda^{\rightarrow} : \lambda^{\rightarrow} \neq 0, \rightarrow \in O_G^*\}$ equals zero in particular excludes such a possibility.[24]

The following corollary presents easy-to-verify sufficient conditions for the requirement in Proposition 9.2, and shows that a robust bipartition equilibrium exists in a large class of networks.

Corollary 9.3 *Suppose that the network G has a maximally independent set $J \subset I$ that satisfies one of the following conditions:*

1) $|J| > |J'|$ *for any independent set* $J' \subset I \setminus J$.

2) $\max_{i \in J} d_i > \max_{i \in I \setminus J} d_i$.

3) $I \setminus J$ *is independent.*

If $h > v^0 - c > 0$, then for the bipartition $(I_1, I_2) = (J, I \setminus J)$, the bipartition equilibrium under h-linearity is robust.

10 Two-sided Markets

An extremely clear-cut characterization of equilibrium pricing under approximate linearity is obtained when we focus on an important class of networks known as bipartite networks. A buyer network G is *bipartite* if there exists a bipartition (I_1, I_2) such that the only links of G are between I_1 and I_2.[25] Any tree, which is a network with no cycle, is bipartite. For example, the line network in Figure 1 is bipartite with the partition $I_1 = \{1, 3\}$ and $I_2 = \{2\}$. More generally, a network is bipartite if and only if the length of every cycle is even. Bipartite networks are the most fundamental class of networks in graph theory.[26] A bipartite network is *complete* if every buyer in I_1 is adjacent to every buyer in I_2. A complete bipartite network is a graph-theoretical representation of a two-sided market with global externalities that receives much attention in the economics literature. For example, we can think of I_1 as the set of sellers and I_2 as the set of buyers of a certain good. In this case, firms A and B are interpreted as the *platforms* that offer marketplace to these sellers and buyers, and their prices are interpreted as *participation fees* required for registration into their platforms. Under global externalities, the value of a platform for an agent on one side of the market is monotonically increasing in the number of agents on the other side of the market who adopt that platform. Our characterization below applies also to two-sided markets with local externalities where the value of a platform to any agent depends on whether the adopters of the platform on the other side are linked to him or not.

[24] If the sequence of indegrees generated by $\rightarrow'$ is the same as (a permutations of) that generated by the reverse of $\rightarrow \in O_G^*$, then $\lambda^{\rightarrow} + \lambda^{\rightarrow'} = 0$.

[25] In other words, both I_1 and I_2 are independent. Condition (3) of Corollary 9.3 corresponds to this case.

[26] See for example Bollobás (1998).

Proposition 10.1 *(Monopolization equilibrium in a two-sided market) Suppose that the buyer network G is bipartite with the bipartition (I_1, I_2). Let h, v^0 and $c \geq 0$ satisfy $h > v^0 - c > 0$. If the externalities v are sufficiently close to h-linear, and $\sum_{i \in I_1} (v^{d_i^2} - v^0) \geq \sum_{i \in I_2} (v^{d_i^1} - v^0)$, then $(p^*, q^*) = (z, z)$ is consistent with a monopolization equilibrium if*

$$z = \begin{cases} v^{d_i^2} - v^0 + c & \text{if } i \in I_1, \\ v^0 - v^{d_i^1} + c & \text{if } i \in I_2. \end{cases}$$

According to Proposition 10.1, the agents on one side are charged markups and those on the other side are offered markdowns. Which side should be offered discounts is determined by the inequality that compares the aggregate externalities that each side enjoys when all agents on the other side adopt the same platform: Markups are charged to the side that enjoys the larger aggregate externalities. This pricing strategy is a natural extension of the bipartition pricing identified in Proposition 7.3 given that the size of a markup/markdown to any buyer (divided by h) approaches the number of his neighbors on the other side in the limit as the externalities approach linearity. Put differently, Proposition 10.1 shows the robustness of bipartition pricing for a bipartite network.

The pricing strategy described in Proposition 10.1 has empirical support. While the theoretical literature finds such a pricing strategy optimal when the platform is a monopoly or when the externalities are global, Proposition 10.1 shows that it is valid even when there is competition and when the externalities are local. In particular, it is the first to identify the exact relationship between the externalities and the direction and size of price discrimination.[27, 28] The literature often discusses global but asymmetric externalities in two-sided markets.[29] In those models, the network effect of the adoption decision of side 1 on side 2 is different from that of side 2 on side 1. Such a model is replicated in the present framework by considering a complete bipartite network with different numbers of buyers on each side. Which side should receive discounts in such markets again depends on the specification of externalities.

Figure 3 illustrates the markups and markdowns specified in Proposition 10.1 when

$$v^4 - v^0 \geq 4(v^1 - v^0), \tag{23}$$

[27]Local externalities are important in two-sided markets as well. Consider, for example, a sport league selling sponsorship rights to firms while selling broadcasting rights to TV stations in different countries. Potential sponsors would be interested in knowing which countries obtain the right given the varying degrees of their interests in those markets.

[28]Alternative explanation of the markup-markdown scheme in two-sided markets is provided by Bolt and Tieman (2008), and Parker and Van Alstyne (2005) among others.

[29]See, for example, Jullien (2011).

so that $I_1 = \{1\}$ and $I_2 = \{2, 3, 4, 5\}$. Buyer 1 at the hub is charged a markup whereas all the buyers in the periphery are offered a discount. We can interpret the discount to the peripheral buyers as a protection against the inducement from the other firm. In fact, when (23) holds, it is relatively more difficult for the other firm, say firm A, to induce the hub buyer to switch: When for example firm B engages in marginal cost pricing $q^c = (c, \ldots, c)$, firm A must pay buyer 1 more than $v^4 - v^0$ to make $x_1 = A$ 1-dominant, whereas he needs to pay just above $4(v^1 - v^0)$ to make $x_i = A$ 1-dominant for all peripheral buyers. When the inequality (23) is reversed, then buyer 1 now receives a discount, whereas the peripheral buyers are charged a markup. Again, which buyer(s) should be protected depends sensitively on the specification of externalities.

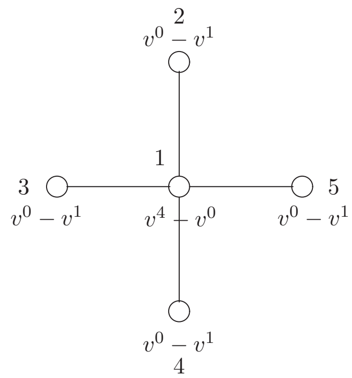

Figure 3. Equilibrium markups and markdowns on a star when $v^4 - v^0 \geq 4(v^1 - v^0)$.

One interesting observation concerns how an equilibrium changes when a link is added to or removed from a network. Suppose that the network G' is obtained from the original bipartite network G by adding a link between buyer i in I_1 and buyer j in I_2. If buyer i is charged a markup and buyer j is offered a markdown in G, the addition of the link typically only affects only the prices charged to buyers i and j: With the new link, the price increases by approximately h for buyer i and decreases by approximately h for buyer j. In some cases, however, the new link may cause the *regime shift* in the pricing strategy. In other words, the addition of a new link may cause I_1 to be the side that receives a discount, and I_2 to be the side that is charged a markup. When $2(v^1 - v^0) + v^4 - v^0 < 3(v^2 - v^0)$ and $(v^1 - v^0) + (v^4 - v^0) > (v^2 - v^0) + (v^3 - v^0)$ in Figure 4, for example, the buyers in I_1 are charged markups in G but offered discounts in G' which is obtained from G by adding a link between 2 and 8.[30]

[30] As seen in this example, the regime shift can take place even when the externalities are marginally decreasing or increasing.

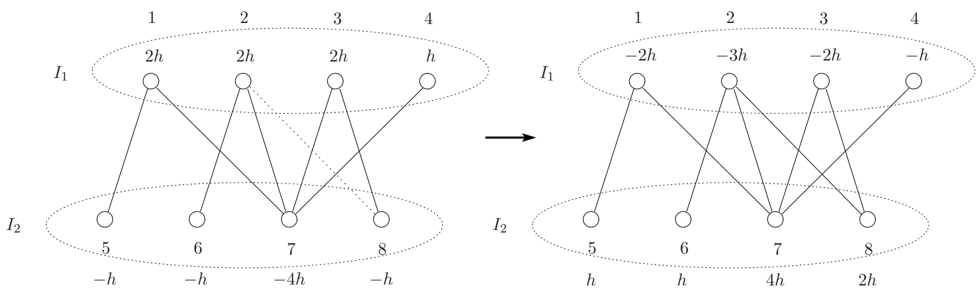

Figure 4. Regime shift with the addition of a link: G (left) and G' (right).

11 Discussion

The essential feature of the market for goods with network externalities is the multiplicity of equilibria. In the present context, this corresponds to the multiplicity of equilibria in the buyers' subgame. Our construction of an equilibrium is based on the assumption that following any deviation by either firm, the buyers coordinate on the extreme equilibrium that least favors the deviator. While this assumption supports the broadest spectrum of equilibrium, it is not consistent with, for example, the assumption that the buyers choose the Pareto efficient alternative whenever there is one.[31] Fundamental multiplicity of equilibria also exists in the pricing game between the firms when the externalities are linear. In this case, any pricing strategy is consistent with an equilibrium as long as the sum of markups and markdowns (divided by the factor of proportion h) it entails for any subset of buyers does not exceed the number of links they have with the rest of the network. Even if we restrict attention to bipartition pricing strategies, there is freedom in the choice of a binary partition as well as in the choice of the partition element to which discounts are offered. On the other hand, the entire set of equilibrium pricing strategies is unknown under non-linear externalities. We have shown that marginal cost pricing is no longer consistent with a monopolization equilibrium except in the two special classes of networks under non-linear externalities. As for bipartition pricing, we have identified conditions under which it is robust and in a bipartite network, have uniquely pinned down the equilibrium pricing strategy under approximate linearity that is "close" to robust bipartition pricing under linearity. These findings may suggest that the number of equilibrium pricing strategies is fewer under non-linear externalities than under linear externalities.

There are a number of interesting extensions of the present model including, for

[31] See Ambrus and Argenziano (2009) and also Jullien (2011). One related issue concerns what happens when one of the firms, say firm A, is *focal* so that the buyers play the A-maximal NE following any price offers. See Jullien (2011).

example, the model with a general number of firms, firms with asymmetric externalities, or a positive degree of compatibility between the goods.[32, 33] It would also be interesting to consider alternatives to the assumptions of perfect price discrimination, public observability of prices, and perfect knowledge of the firms about the network.[34]

Appendix

Proof of Proposition 5.1 We show that x^A is a NE. It will be then clear that x^A is a A-maximal NE since any NE y must be rationalizable (*i.e.*, $y \in S^K$), and since $x_i^A = A$ whenever A is rationalizable. For buyer $i \in D$, x_i^A is dominant and hence is optimal. If i has no dominant action ($i \notin D$) and $x_i = A$ is rationalizable, then $x_i^A = A$ by definition. If

$$u_i\left(x^A\right) < 0 = u_i\left(x_i = \emptyset, x_{-i}^A\right),$$

then $\emptyset$ would be rationalizable and dominate $x_i = A$. If

$$u_i\left(x^A\right) < u_i\left(x_i = B, x_{-i}^A\right),$$

then B would be rationalizable and dominate $x_i = A$ since buyer $j \neq i$ plays $x_j^A = B$ only if $x_j = B$ is iteratively dominant and plays $x_j^A = A$ whenever A is rationalizable. In either case, we would have $A \notin S_i^{K+1} \subsetneq S_i^K$, a contradiction.

If i has no dominant action ($i \notin D$) and $x_i = A$ is not rationalizable, then the set of rationalizable actions equals $S_i^K = \{B, \emptyset\}$ and $x_i^A = \emptyset$ by definition. If

$$0 = u_i\left(x^A\right) < u_i\left(x_i = A, x_{-i}^A\right),$$

then A would be rationalizable, a contradiction. If

$$0 = u_i\left(x^A\right) < u_i\left(x_i = B, x_{-i}^A\right),$$

then $x_i = B$ would be rationalizable and dominate $x_i = \emptyset$ since buyer $j \neq i$ plays $x_j^A = B$ only if it is iteratively dominant. This is a contradiction to the fact that $x_i = \emptyset$ is rationalizable. □

[32] When there is small asymmetry in the marginal cost, the monopolization equilibrium survives with the more efficient firm monopolizing the market.

[33] Endogenous determination of compatibility levels by the firms is one topic that has received much attention in the literature. For example, Baake and Boom (2001) find in their model of global network externalities that the firms always choose to offer compatibility in equilibrium.

[34] Price discrimination in the form of one full price and one discount price is studied in Candogan *et al.* (2012). Pasini *et al.* (2008) formulate a two-sided market model in which firms only know the degree distribution of the buyers.

Proof of Lemma 6.2. Denote $D_A^k = D_A^k(p, q)$ for $k = 1, \ldots, K$, and suppose that adjacent buyers 1 and 2 both find $x_1 = x_2 = A$ k-dominant (*i.e.*, $1,\ 2 \in D_A^k(p, q)$) for some $k \le K$. Then it follows from (9) that for $i = 1,\ 2$,

$$p_i < \min \left\{ v^{\alpha_i^k(p,q)} - v^{\beta_i^k(p,q)} + q_i,\ v^{\alpha_i^k(p,q)} \right\}.$$

Consider now p' such that $p'_i = p_i$ for $i \ne 2$, and

$$p_2 < p'_2 < \min \left\{ v^{\alpha_2^k(p,q)+1} - v^{\beta_2^k(p,q)} + q_2, v^{\alpha_2^k(p,q)+1} \right\}. \tag{24}$$

We then have

$$D_A^\ell(p', q) = D_A^\ell(p, q) \text{ for } \ell \le k-1, \quad \text{and} \quad D_A^k(p', q) = D_A^k(p, q) \setminus \{2\}.$$

Since 1 is adjacent to 2 ($1 \in N_2$), in round $k+1$ of the iteration process under (p', q), we have,

$$\alpha_2^{k+1}(p', q) = \left| \left\{ j \in N_2 : x_j = A \text{ is } k\text{-dominant under } (p', q) \right\} \right| = \alpha_2^k(p, q) + 1,$$

and

$$\beta_2^{k+1}(p', q) = \left| \left\{ j \in N_2 : x_j = B \text{ is } k\text{-rationalizable under } (p', q) \right\} \right| \le \beta_2^k(p, q).$$

It then follows from (24) that

$$p'_2 < \min \left\{ v^{\alpha_2^{k+1}(p',q)} - v^{\beta_2^{k+1}(p',q)} + q_2, v^{\alpha_2^{k+1}(p',q)} \right\}.$$

so that $x_2 = A$ is $(k+1)$-dominant ($2 \in D_A^{k+1}(p', q)$) by (9). This further implies that $D_A(p', q) = D_A(p, q)$. Hence

$$\pi_A(p', q, \sigma^B) = \sum_{i \in D_A(p',q)} (p'_i - c) > \sum_{i \in D_A(p,q)} (p_i - c) = \pi_A(p, q, \sigma^B),$$

implying that p' is a strictly better response to (q, σ^B) than p. $\square$

Proof of Proposition 6.1. If (p^*, q^*, σ) is an SPE for σ which is extremal with respect to (p^*, q^*), then (p^*, q^*) is clearly an SPE price vector. Conversely, suppose that $(p^*, q^*, \hat{\sigma})$ is an equilibrium for some strategy profile $\hat{\sigma}$ of the buyers. Let σ be the buyers' strategy profile which (i) is extremal with respect to (p^*, q^*), and (ii) chooses the same action profile as $\hat{\sigma}$ on the path: $\sigma(p^*, q^*) = \hat{\sigma}(p^*, q^*)$. To show that (p^*, q^*, σ) is an equilibrium, we derive a contradiction by supposing that firm A has a profitable deviation p from p^* against (q^*, σ):

$$\pi_A(p, q^*, \sigma) > \pi_A(p^*, q^*, \sigma). \tag{25}$$

By the definition of σ, only those buyers who have A as their iteratively dominant action choose A: $\{i : \sigma_i(p, q^*) = A\} = D_A(p, q^*)$. Define an alternative price vector p' of firm A as follows:

$$p'_i = \begin{cases} p_i & \text{if } i \in D_A(p, q^*), \\ c & \text{otherwise.} \end{cases}$$

We will show that if $x_i = A$ is iteratively dominant for buyer i under (p, q^*), then it is also iteratively dominant under (p', q^*): $D_A(p', q^*) \supset D_A(p, q^*)$. We will show this by proving that $\cup_{\ell=1}^k D_A^\ell(p', q^*) \supset \cup_{\ell=1}^k D_A^\ell(p, q^*)$ and $R_B^k(p', q^*) \subset R_B^k(p, q^*)$ for each $k = 1, \ldots, K$, where K is the number of the maximum number of iterations under (p, q^*), and R_B^k is the set of buyers for whom $x_i = B$ is k-rationalizable as defined in (8). Suppose first that $k = 1$. We have

$$i \in D_A^1(p, q^*) \quad \Leftrightarrow \quad v^0 - p_i > \max\{v^{d_i} - q_i^*, 0\}, \tag{26}$$

$$i \in D_A^1(p', q^*) \quad \Leftrightarrow \quad v^0 - p'_i > \max\{v^{d_i} - q_i^*, 0\}. \tag{27}$$

If $i \in D_A^1(p, q^*)$, then $p'_i = p_i$ and hence (27) holds. It follows that $D_A^1(p, q^*) \subset D_A^1(p', q^*)$. On the other hand,

$$i \in R_B^1(p', q^*) \quad \Leftrightarrow \quad v^{d_i} - q_i^* \geq \max\{v^0 - p'_i, 0\}, \tag{28}$$

$$i \in R_B^1(p, q^*) \quad \Leftrightarrow \quad v^{d_i} - q_i^* \geq \max\{v^0 - p_i, 0\}. \tag{29}$$

If $i \in R_B^1(p', q^*)$ and $i \in D_A(p, q^*)$, (29) holds since then $p'_i = p_i$. Let $i \in R_B^1(p', q^*)$ and $i \notin D_A(p, q^*)$. If $v^0 - p_i > 0$, then $v^{d_i} - q_i^* \geq v^0 - p_i$ must hold since otherwise, $x_i = A$ would be 1-dominant for i under (p, q^*), a contradiction. If $v^0 - p_i \leq 0$, then $v^{d_i} - q_i^* \geq 0$ clearly holds by (28). It follows that (29) holds in both cases so that $R_B^1(p', q^*) \subset R_B^1(p, q^*)$.

As an induction hypothesis, suppose that $\cup_{\kappa=1}^\ell D_A^\kappa(p, q^*) \subset \cup_{\kappa=1}^\ell D_A^\kappa(p', q^*)$ and $R_B^\ell(p, q^*) \supset R_B^\ell(p', q^*)$ for $\ell = 1, \ldots, k-1$ for some $k \geq 2$. We then have $\tilde{\alpha}_i^k \equiv \alpha_i^k(p', q^*) \geq \alpha_i^k(p, q^*) \equiv \alpha_i^k$ and $\tilde{\beta}_i^k \equiv \beta_i^k(p', q^*) \leq \beta_i^k(p, q^*) \equiv \beta_i^k$ for every i. Note that

$$i \in \cup_{\kappa=1}^k D_A^\kappa(p, q^*) \quad \Leftrightarrow \quad v^{\alpha_i^k} - p_i > \max\{v^{\beta_i^k} - q_i^*, 0\}, \tag{30}$$

$$i \in \cup_{\kappa=1}^k D_A^\kappa(p', q^*) \quad \Leftrightarrow \quad v^{\tilde{\alpha}_i^k} - p'_i > \max\{v^{\tilde{\beta}_i^k} - q_i^*, 0\}. \tag{31}$$

Take any $i \in \cup_{\kappa=1}^k D_A^\kappa(p, q^*)$. Given that $p'_i = p_i$ for any such i, (31) holds since $\tilde{\alpha}_i^k \geq \alpha_i^k$ and $\tilde{\beta}_i^k \leq \beta_i^k$. It follows that $i \in \cup_{\kappa=1}^k D_A^\kappa(p', q^*)$. We hence conclude that $\cup_{\kappa=1}^k D_A^\kappa(p, q^*) \subset \cup_{\kappa=1}^k D_A^\kappa(p', q^*)$. On the other hand,

$$i \in R_B^k(p', q^*) \quad \Leftrightarrow \quad v^{\tilde{\beta}_i^k} - q_i^* \geq \max\{v^{\tilde{\alpha}_i^k} - p'_i, 0\}, \tag{32}$$

$$i \in R_B^k(p, q^*) \quad \Leftrightarrow \quad v^{\beta_i^k} - q_i^* \geq \max\{v^{\alpha_i^k} - p_i, 0\}. \tag{33}$$

If $i \in R_B^k(p', q^*)$ and $i \in D_A(p, q^*)$, then $p'_i = p_i$ so that (32) implies (33) since $\tilde{\alpha}_i^k \geq \alpha_i^k$ and $\tilde{\beta}_i^k \leq \beta_i^k$. Let $i \in R_B^k(p', q^*)$ and $i \notin D_A(p, q^*)$. If $v^{\alpha_i^k} - p_i > 0$, then

$v^{\beta_i^k} - q_i^* \geq v^{\alpha_i^k} - p_i$ must hold since otherwise, $x_i = A$ would be k-dominant for i under (p, q^*), a contradiction. If $v^{\alpha_i^k} - p_i \leq 0$, then $v^{\beta_i^k} - q_i^* \geq v^{\tilde{\beta}_i^k} - q_i^* \geq 0$ holds by (32). It follows that (33) holds in both cases so that $R_B^k(p', q^*) \subset R_B^k(p, q^*)$. This completes the induction step.

The above argument shows that $\pi_A(p', q^*, \sigma) = \pi_A(p, q^*, \sigma)$ since any buyer $i \in D_A(p, q^*)$ chooses A also under (p', q^*) for the same price $p_i' = p_i$, and the contribution to firm A's payoff of any buyer $i \in D_A(p', q^*) \setminus D_A(p, q^*)$ equals zero since $p_i' = c$. Finally, $\{i : \hat{\sigma}_i(p', q^*) = A\} \supset D_A(p', q^*)$, and if $\hat{\sigma}_i(p', q^*) = A$ and $i \notin D_A(p', q^*)$, then i's contribution to firm A's payoff equals zero since $p_i' = c$. It hence follows that $\pi_A(p', q^*, \hat{\sigma}) = \pi_A(p', q^*, \sigma) = \pi_A(p, q^*, \sigma)$. This, along with $\pi_A(p^*, q^*, \hat{\sigma}) = \pi_A(p^*, q^*, \sigma)$ by the definition of σ, shows that if (25) holds, then p' is a profitable deviation for firm A against $(q^*, \hat{\sigma})$, which is a contradiction. □

Proof of Lemma 6.3. Let $\varepsilon > 0$ and orientation $\rightarrow \in O_G$ be given. Define the price vector $p = (p_i)_{i \in I}$ by

$$p_i = \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\} - \varepsilon. \tag{34}$$

p is then a DC vector so that $D_A(p, q^*) = I$. Hence, firm A's payoff under (p, q^*, σ) satisfies

$$\pi_A(p, q^*, \sigma) = \sum_{i \in I} \Big(\min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\} - \varepsilon - c\Big).$$

Since $\varepsilon > 0$ and $\rightarrow \in O_G$ are arbitrary, if (16) does not hold, then we would have a contradiction $\pi_A(p, q^*, \sigma) > \pi_A(p^*, q^*, \sigma)$. □

Proof of Proposition 7.1. Let $p^* = q^* = (c, \ldots, c)$. Let σ be the extremal strategy profile with respect to (p^*, q^*) with $\sigma(p^*, q^*) = (B, \ldots, B)$ so that B monopolizes the market under (p^*, q^*). Now consider any deviation $p \neq p^*$ by firm A and write $D_A^k = D_A^k(p, q^*)$ $(k = 1, \ldots, K)$. Since $v^0 - q_i^* = v^0 - c \geq 0$, $x_i = B$ is never dominated by $x_i = \emptyset$. Furthermore, if $x_i = A$ dominates $x_i = B$, then $x_i = A$ also dominates $x_i = \emptyset$. If follows that for any i and k, either $x_i = A$ is the unique $(k-1)$-rationalizable action ($S_i^{k-1} = \{A\}$) or $x_i = B$ is $(k-1)$-rationalizable ($B \in S_i^{k-1}$). Hence, for any i and k,

$$\beta_i^k = d_i - \alpha_i^k.$$

It then follows from (9) that

$$p_i < \min\left\{v^{\alpha_i^k} - v^{\beta_i^k} + q_i^*, v^{\alpha_i^k}\right\} \leq v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + c.$$

Denote the number of links in $J \subset I$ by $L(J)$, and that between J and $J' \subset I \setminus J$ by $L(J, J')$. We then have

$$\sum_{i \in D_A} d_i = 2L(D_A) + L(D_A, I \setminus D_A) \quad \text{and} \quad \sum_{i \in D_A^k} \alpha_i^k = L(D_A). \tag{35}$$

Hence, firm A's payoff satisfies

$$\begin{aligned}\pi_A(p, q^*, \sigma^B) &= \sum_{i \in D_A} (p_i - c) \\ &\leq \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} \right) \\ &= h \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(2\alpha_i^k - d_i \right) \\ &= -hL(D_A, I \setminus D_A) \leq 0.\end{aligned}$$

Therefore, p is not a profitable deviation. □

Proof of Proposition 7.2 The proof uses three lemmas (Lemmas A.1, A.2, and A.3) presented below.

1) Suppose that (18) holds. Let σ be extremal with respect to (p^*, q^*) and $\sigma(p^*, q^*) = (B, \dots, B)$. It follows from Lemma A.2 that if p is any profitable deviation, then $\beta_i^k = d_i - \alpha_i^k$ for any i and k. Hence, if $i \in D_A(p, q^*)$, p_i must satisfy

$$p_i < \min \left\{ v^{\alpha_i^k} - v^{\beta_i^k} + q_i^*,\, v^{\alpha_i^k} \right\} \leq v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*. \tag{36}$$

Take any $J \subset I$ and take any p such that $D_A(p, q^*) = J$. It then follows from Lemma A.1 that

$$\begin{aligned}\pi_A(p, q^*, \sigma) &= \sum_{i \in J} (p_i - c) \\ &< \sum_{i \in J} (v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* - c) \\ &\leq -hL(J, I \setminus J) + hL(J, I \setminus J) = 0.\end{aligned}$$

Hence, firm A has no profitable deviation.

2) Suppose that (18) fails. Suppose that firm B monopolizes the market under (p^*, q^*): $\sigma(p^*, q^*) = (B, \dots, B)$. In view of Proposition 6.1, we may suppose that σ is extremal with respect to (p^*, q^*). If $\sum_{i \in I} (z_i - c) = \pi_B(p^*, q^*, \sigma) < 0$, then (p^*, q^*) is clearly inconsistent with an SPE. Suppose then that $\sum_{i \in J} (z_i - c) > hL(J, I \setminus J)$ for some $J \subset I$. By setting $K = \emptyset$ in Lemma A.3, we see that when the buyers play the B-maximal NE σ^B, firm A can choose p so that $D_A(p, q^*) = J$ and make its payoff arbitrarily close to

$$r(J \mid \emptyset, q^*) \geq -hL(J, I \setminus J) + \min \Big\{ \sum_{j \in J} (q_j^* - c),\, n(v^0 - c) + hL(J, I \setminus J) \Big\}. \tag{37}$$

By assumption, the RHS of (37) is > 0, implying that there exists p such that $\pi_A(p, q^*, \sigma^B) > 0$. Since σ is extremal with respect to (p^*, q^*), we have $\pi_A(p, q^*, \sigma) = \pi_A(p, q^*, \sigma^B) > 0 = \pi_A(p^*, q^*, \sigma)$ so that p is a profitable deviation against q^*. $\square$

Lemma A.1 *Let the network G be given and suppose that the externalities are h-linear for $h > 0$. Then for any (p, q) and $D_A = D_A(p, q)$,*

$$\frac{1}{h}\sum_{k=1}^{K}\sum_{i\in D_A^k}\left(v^{\alpha_i^k} - v^{d_i-\alpha_i^k}\right) = -L(D_A, I\setminus D_A).$$

Proof of Lemma A.1 The conclusion follows since $v^{\alpha_i^k} - v^{d_i-\alpha_i^k} = h(2\alpha_i^k - d_i)$, $\sum_{k=1}^{K}\sum_{i\in D_A^k}\alpha_i^k = L(D_A)$, and $\sum_{k=1}^{K}\sum_{i\in D_A^k} d_i = 2L(D_A) + L(D_A, I\setminus D_A)$. $\square$

Lemma A.2 *Let the buyer network G be given. Suppose that the externalities v are h-linear for $h > 0$. Let q^* be any price vector satisfying*

$$\sum_{i\in J}(q_i^* - c) \le hL(J, I\setminus J) \text{ for any } J \subset I. \tag{38}$$

If firm A's price vector p is such that $\pi_A(p, q^*, \sigma^B) \ge 0$, then the following holds under (p, q^*): For any $i \in I$ and $k = 1, \dots, K$, either

$$\begin{aligned}&(i)\ x_i = B \text{ is } k\text{-rationalizable (i.e., } B \in S_i^k\text{), or}\\ &(ii)\ x_i = A \text{ is the unique } k\text{-rationalizable action (i.e., } \{A\} = S_i^k\text{).}\end{aligned} \tag{39}$$

Furthermore, for any such p, $\alpha_i^k + \beta_i^k = d_i$ for every i and k.

Proof of Lemma A.2 Note that the second statement of the proposition follows from (39): For any $i \in I$ and $k \ge 2$, every neighbor j of i has either $x_j = A$ as their unique $(k-1)$-rationalizable action or $x_j = B$ as one of the $(k-1)$-rationalizable actions. Hence, $\alpha_i^k + \beta_i^k = d_i$.

Write $D_A^k \equiv D_A^k(p, q^*)$. We show that the failure of (39) implies $\pi_A(p, q^*, \sigma^B) < 0$. For simplicity, we suppose that (39) fails for a single i so that $B \notin S_i^k$ and $\{A\} \ne S_i^k$ for some k. Let $k \ge 2$ be the smallest such k so that $B \in S_i^{k-1}$ and $B \notin S_i^k$. We must have $v^{\beta_i^k} < q_i^*$ since if $v^{\beta_i^k} \ge q_i^*$, for $B \notin S_i^k$ to take place, $x_i = B$ must be dominated by $x_i = A$ in S^{k-1} so that $v^{\alpha_i^k} - p_i > v^{\beta_i^k} - q_i^* \ge 0$. This, however, shows that $x_i = A$ also dominates $x_i = \emptyset$ and hence is the unique k-rationalizable action ($S_i^k = \{A\}$), which is a contradiction. For any $j \ne i$, (39) holds by assumption, and hence for any $\ell = 1, \dots, K$,

$$j \in D_A^\ell \Rightarrow S_j^{\ell-1} \ne \{A\} \Rightarrow B \in S_j^{\ell-1}.$$

Suppose first that $i \in D_A^m$ for some $m \geq k$. By the above observation, for any j and ℓ such that $j \in D_A^\ell$, β_j^ℓ is given as follows:

$$\beta_j^\ell = \begin{cases} d_j - \alpha_j^\ell & \text{if } j \in D_A^\ell \setminus N_i, \text{ or} \\ & \text{if } j \in N_i \cap D_A^\ell \text{ and } \ell \leq k \text{ or } \ell \geq m+1, \\ d_j - \alpha_j^\ell - 1 & \text{if } j \in N_i \cap D_A^\ell \text{ and } k+1 \leq \ell \leq m-1. \end{cases} \tag{40}$$

Furthermore, since $\beta_i^m \leq \beta_i^k$, if $x_i = A$ is m-dominant, then p_i should satisfy

$$p_i < \min\left\{v^{\alpha_i^m} - v^{\beta_i^m} + q_i^*,\, v^{\alpha_i^m}\right\} = v^{\alpha_i^m}. \tag{41}$$

Using (40) and (41), we can evaluate firm A's payoff under (p, q^*, σ^B) as follows:

$$\begin{aligned} \pi_A(p, q^*, \sigma^B) &= \sum_{j \in D_A} (p_j - c) \\ &< \sum_{\ell=1}^{K} \sum_{j \in D_A^\ell} \left(\min\left\{v^{\alpha_j^\ell} - v^{\beta_j^\ell} + q_j^*,\, v^{\alpha_j^\ell}\right\} - c\right) \\ &\leq \sum_{\ell=k+1}^{m-1} \sum_{j \in N_i \cap D_A^\ell} \left(v^{\alpha_j^\ell} - v^{d_j - \alpha_j^\ell - 1} + q_j^* - c\right) \\ &\quad + \sum_{\substack{\ell \leq k \\ \ell \geq m+1}} \sum_{j \in N_i \cap D_A^\ell} \left(v^{\alpha_j^\ell} - v^{d_j - \alpha_j^\ell} + q_j^* - c\right) \\ &\quad + \sum_{\ell=1}^{K} \sum_{\substack{j \neq i \\ j \in D_A^\ell \setminus N_i}} \left(v^{\alpha_j^\ell} - v^{d_j - \alpha_j^\ell} + q_j^* - c\right) \\ &\quad + \left(v^{\alpha_i^m} - c\right) \end{aligned} \tag{42}$$

Rewriting of (42) yields

$$\begin{aligned} \pi_A(p, q^*, \sigma^B) &< \sum_{\ell=1}^{K} \sum_{j \in D_A^\ell} \left(v^{\alpha_j^\ell} - v^{d_j - \alpha_j^\ell} + q_j^* - c\right) \\ &\quad + \sum_{\ell=k+1}^{m-1} \sum_{j \in N_i \cap D_A^\ell} \left(v^{d_j - \alpha_j^\ell} - v^{d_j - \alpha_j^\ell - 1}\right) + v^{d_i - \alpha_i^m} - q_i^* \\ &\leq \sum_{\ell=k+1}^{m-1} \sum_{j \in N_i \cap D_A^\ell} \left(v^{d_j - \alpha_j^\ell} - v^{d_j - \alpha_j^\ell - 1}\right) + v^{d_i - \alpha_i^m} - q_i^*, \end{aligned} \tag{43}$$

where the last inequality follows from Lemma A.1 and (38). Linearity then implies that

$$\begin{aligned}\pi_A(p,q^*,\sigma^B) &< v^{d_i-\alpha_i^m} - q_i^* + \sum_{\ell=k+1}^{m-1} \sum_{j\in N_i\cap D_A^\ell} h \\ &= v^0 + h(d_i-\alpha_i^m) - q_i^* + h\left|N_i \cap \cup_{\ell=k+1}^{m-1} D_A^\ell\right|.\end{aligned} \tag{44}$$

When $m-1 \geq k+1$, (44) reduces to

$$\begin{aligned}\pi_A(p,q^*,\sigma^B) &< v^0 + h(d_i-\alpha_i^m) - q_i^* + h(\alpha_i^m - \alpha_i^{k+1}) \\ &= v^0 + h(d_i - \alpha_i^{k+1}) - q_i^* \\ &= v^{\beta_i^{k+1}} - q_i^* \\ &\leq v^{\beta_i^k} - q_i^* < 0.\end{aligned} \tag{45}$$

The same inequality holds true also when $m = k+1$ or $m = k$ since $\alpha_i^m = \alpha_i^{k+1}$ in those cases.[35] When $i \notin D_A$, (42) holds with m replaced by $K+1$ and without the last term $v^{\alpha_i^m} - c \geq 0$ on the right-hand side. It follows that $\pi_A(p,q^*,\sigma^B) < 0$ in this case as well. □

Suppose that the externalities are h-linear. Given firm B's price vector q and disjoint sets K, $J \subset I$ of buyers, suppose that firm A offers p such that under (p,q), (i) $x_i = A$ is iteratively dominant for $i \in K \cup J$, and (ii) the buyers in K precede the buyers in J. Let $r_A(J \mid K,q)$ denote the supremum of firm A's payoff from J when it chooses a price vector p satisfying these conditions:

$$r_A(J \mid K,q) = \sup\left\{\sum_{i\in J}(p_i - c) : K\cup J = D_A(p,q),\ i \prec j \text{ if } i \in K \text{ and } j \in J\right\}.$$

Lemma A.3 *For any K and $J \subset I$ such that $K \cap J = \emptyset$, if we denote $\bar{q}_i = \min\{q_i, v^{d_i - d_i^K}\}$, then*

$$\begin{aligned}r(J \mid K,q) \geq\ & h\Big\{L(J,K) - L(J, I\setminus(K\cup J))\Big\} \\ &+ \min\Big\{\sum_{j\in J}(\bar{q}_j - c),\ n(v^0 - c) + hL(J, I\setminus(K\cup J))\Big\}.\end{aligned} \tag{46}$$

Proof. We proceed by induction on the size of J. Take any J such that $J = \{j\}$ for any $j \in I$. Then j finds A dominant and is preceded by buyers in K if $v^{d_j^K} - p_j >$

[35] When $m = k$, $\alpha_i^{k+1} = \alpha_i^{m+1} = \left|N_i \cap \cup_{\ell=1}^m D_A^\ell\right| = \left|N_i \cap \cup_{\ell=1}^{m-1} D_A^\ell\right| = \alpha_i^m$ since $i \in D_A^m$ implies $N_i \cap D_A^m = \emptyset$ by Lemma 6.2.

$\max\{v^{d_j - d_j^K} - q_j, 0\}$ so that any p_j with $p_j < v^{d_j^K} - v^{d_j - d_j^K} + \min\{q_j, v^{d_j - d_j^K}\} = v^{d_j^K} - v^{d_j - d_j^K} + \bar{q}_j$ will ensure that $j \in D_A(p, q)$. Hence, (46) holds since

$$\begin{aligned}
&r(\{j\} \mid K, q) \\
&\geq v^{d_j^K} - v^{d_j - d_j^K} + \bar{q}_j - c \\
&= h\Big\{L(\{j\}, K) - L(\{j\}, I \setminus (K \cup \{j\}))\Big\} \\
&+ \min\Big\{\bar{q}_j - c,\, v^0 - c + hL(\{j\}, I \setminus (K \cup \{j\}))\Big\}.
\end{aligned}$$

Let $n \geq 1$ be given, and suppose as an induction hypothesis that (46) holds for every J such that $|J| \leq n$. Take any J such that $|J| = n + 1$. Suppose without loss of generality that $1 \in J$ and let $J_{-1} = J \setminus \{1\}$. Note that $r(J \mid K, q)$ satisfies

$$\begin{aligned}
r(J \mid K, q) \geq \max\Big\{&r(\{1\} \mid K, q) + r(J_{-1} \mid K \cup \{1\}, q), \\
&r(J_{-1} \mid K, q) + r(\{1\} \mid K \cup J_{-1}, q)\Big\}.
\end{aligned} \tag{47}$$

The first term on the right-hand side equals the bound when buyer 1 precedes buyers in J_{-1}, and the second term equals the bound when buyers in J_{-1} precede buyer 1. By the induction hypothesis, we can evaluate each term on the right-hand side of (47) as follows. First, since

$$r(\{1\} \mid K, q) \geq h\Big\{L(\{1\}, K) - L(\{1\}, I \setminus (K \cup \{1\}))\Big\} + \bar{q}_1 - c,$$

and

$$\begin{aligned}
r(J_{-1} \mid K \cup \{1\}, q) \geq h\Big\{&L(J_{-1}, K \cup \{1\}) - L(J_{-1}, I \setminus (K \cup J))\Big\} \\
&+ \min\Big\{\sum_{j \in J_{-1}} (\bar{q}_j - c),\, n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J))\Big\},
\end{aligned}$$

we have

$$\begin{aligned}
&r(J \mid K, q) - h\Big\{L(J, K) - L(J, I \setminus (K \cup J))\Big\} \\
&\geq \bar{q}_1 - c + \min\Big\{\sum_{j \in J_{-1}} (\bar{q}_j - c),\, n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J))\Big\}.
\end{aligned} \tag{48}$$

Next, since

$$\begin{aligned}
r(J_{-1} \mid K, q) \geq h\Big\{&L(J_{-1}, K) - L(J_{-1}, I \setminus (K \cup J_{-1}))\Big\} \\
&+ \min\Big\{\sum_{j \in J_{-1}} (\bar{q}_j - c),\, n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J_{-1}))\Big\},
\end{aligned}$$

and

$$r(\{1\} \mid K \cup J_{-1}, q) \geq h\Big\{L(\{1\}, K \cup J_{-1}) - L(\{1\}, I \setminus (K \cup J))\Big\} \\ + \min\Big\{\bar{q}_1 - c,\ v^0 - c + hL(\{1\}, I \setminus (K \cup J))\Big\},$$

we have

$$\begin{aligned} & r(J \mid K, q) - h\Big\{L(J, K) - L(J, I \setminus (K \cup J))\Big\} \\ & \geq \min\Big\{\sum_{j \in J_{-1}} (\bar{q}_j - c),\ n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J_{-1}))\Big\} \\ & + \min\Big\{\bar{q}_1 - c,\ v^0 - c + hL(\{1\}, I \setminus (K \cup J))\Big\}. \end{aligned} \tag{49}$$

We now consider the following two cases separately. Let $\hat{d}_j = |N_j \setminus (K \cup J)|$ be the number of j's neighbors not in $K \cup J$.

Case 1. If $\bar{q}_j \leq v^{\hat{d}_j}$ for every $j \in J$, then

$$\sum_{j \in J_{-1}} (\bar{q}_j - c) \leq n(v^0 - c) + h \sum_{j \in J_{-1}} \hat{d}_j = n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J)),$$

Substitution of this into (48) yields

$$\begin{aligned} r(J \mid K, q) - h\Big\{L(J, K) - L(J, I \setminus (K \cup J))\Big\} & \geq \bar{q}_1 - c + \sum_{j \in J_{-1}} (\bar{q}_j - c) \\ & = \sum_{j \in J} (\bar{q}_j - c). \end{aligned}$$

Case 2. If $\bar{q}_j > v^{\hat{d}_j}$ for some $j \in J$, suppose without loss of generality that $\bar{q}_1 > v^{\hat{d}_1}$ so that

$$\bar{q}_1 - c > v^0 - c + \hat{d}_1 h.$$

First, if $\sum_{j \in J_{-1}} (\bar{q}_j - c) > n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J_{-1}))$, then it follows from (49) that

$$\begin{aligned} & r(J \mid K, q) - h\Big\{L(J, K) - L(J, I \setminus (K \cup J))\Big\} \\ & \geq n(v^0 - c) + L(J_{-1}, I \setminus (K \cup J_{-1}))h + v^0 - c + \hat{d}_1 h \\ & = (n+1)(v^0 - c) + h\Big\{L(J_{-1}, I \setminus (K \cup J_{-1})) + L(\{1\}, I \setminus (K \cup J))\Big\} \\ & = (n+1)(v^0 - c) + h\Big\{L(J, I \setminus (K \cup J)) + L(J_{-1}, \{1\})\Big\} \\ & \geq (n+1)(v^0 - c) + hL(J, I \setminus (K \cup J)). \end{aligned}$$

Next, if $n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J)) < \sum_{j \in J_{-1}} (\bar{q}_j - c) \leq n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J_{-1}))$, then it follows from (48) that

$$\begin{aligned}
&r(J \mid K, q) - h\Big\{L(J, K) - L(J, I \setminus (K \cup J))\Big\} \\
&\geq \bar{q}_1 - c + n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J)) \\
&> v^0 - c + \hat{d}_1 h + n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J)) \\
&= (n+1)(v^0 - c) + h\Big\{L(J_{-1}, I \setminus (K \cup J)) + L(\{1\}, I \setminus (K \cup J))\Big\} \\
&= (n+1)(v^0 - c) + hL(J, I \setminus (K \cup J)).
\end{aligned}$$

Finally, if $\sum_{j \in J_{-1}} (\bar{q}_j - c) \leq n(v^0 - c) + hL(J_{-1}, I \setminus (K \cup J))$, then it follows from (48) that

$$\begin{aligned}
&r(J \mid K, q) - h\Big\{L(J, K) - L(J, I \setminus (K \cup J))\Big\} \\
&\geq \bar{q}_1 - c + \sum_{j \in J_{-1}} (\bar{q}_j - c) = \sum_{j \in J} (\bar{q}_j - c).
\end{aligned}$$

In all cases, we have shown that (46) holds for any J with $|J| = n+1$. This completes the proof. □

Proof of Proposition 7.3. The proof uses Lemma A.4 presented below. Take any bipartition (I_1, I_2) of the buyer set and let z be the corresponding bipartition price vector. Define $p^* = q^* = z$ and σ to be extremal with respect to (p^*, q^*) with $\sigma(p^*, q^*) = (B, \ldots, B)$. Suppose that $p \neq p^*$ is an arbitrary deviation by firm A. By Lemma A.2, we may restrict attention to p such that

$$\beta_i^k(p, q^*) = d_i - \alpha_i^k(p, q^*) \text{ for any } i \text{ and } k.$$

If $i \in D_A$, then (9) implies that

$$p_i < \min \left\{ v^{\alpha_i^k} - v^{\beta_i^k} + q_i^*, v^{\alpha_i^k} \right\} \leq v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*.$$

Furthermore, Lemma A.4 shows that

$$\frac{1}{h} \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* - c \right) = -L(D_A, I_1 \setminus D_A) - L(I_2 \cap D_A, I \setminus D_A) \leq 0.$$

It hence follows that

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^{K} \sum_{i \in D_A^k} (p_i - c) < \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* - c \right) \leq 0.$$

Therefore, no deviation p is profitable. □

Lemma A.4 *Let the network G be given and suppose that the externalities are h-linear for $h > 0$. If (I_1, I_2) is any bipartition and $q^* = z$ for the bipartition price vector z given (I_1, I_2) defined in (19), then for any p,*

$$\frac{1}{h}\sum_{k=1}^{K}\sum_{i\in D_A^k}\left(v^{\alpha_i^k} - v^{d_i-\alpha_i^k} + q_i^* - c\right) = -L(D_A, I_1 \setminus D_A) - L(I_2 \cap D_A, I \setminus D_A).$$

Proof. It follows from Lemma A.1 and the definition of q^* that

$$\begin{aligned}&\frac{1}{h}\sum_{k=1}^{K}\sum_{i\in D_A^k}\left(v^{\alpha_i^k} - v^{d_i-\alpha_i^k} + q_i^* - c\right)\\ &= -L(D_A, I \setminus D_A) + \sum_{i\in D_A\cap I_1} d_i^2 - \sum_{i\in D_A\cap I_2} d_i^1.\end{aligned} \tag{50}$$

Since

$$\sum_{i\in D_A\cap I_1} d_i^2 - \sum_{i\in D_A\cap I_2} d_i^1 = L(I_1\cap D_A, I_2\setminus D_A) - L(I_2\cap D_A, I_1\setminus D_A), \tag{51}$$

and

$$\begin{aligned}L(D_A, I\setminus D_A) = {}& L(I_1\cap D_A, I_1\setminus D_A) + L(I_1\cap D_A, I_2\setminus D_A)\\ &+ L(I_2\cap D_A, I\setminus D_A),\end{aligned}$$

the right-hand side of (50) reduces to $-L(D_A, I_1 \setminus D_A) - L(I_2 \cap D_A, I \setminus D_A)$. □

Proof of Lemma 8.1. 1) Suppose that G is neither cyclic nor complete. We will consider the following two cases separately.

i) G is not regular.

Since G is not regular, we may suppose, with renaming of buyers if necessary, that buyers 1 and 2 are adjacent, and that $d_1 = \bar{d}$ and $d_2 < \bar{d}$, where $\bar{d} \geq 2$ is the highest degree in G. Suppose further that buyer 3 is adjacent to 1 but not to 2. To see that there exists such a buyer, suppose to the contrary that every neighbor of 1 (except 2) is also a neighbor of 2. Then 2 has at least $\bar{d}$ neighbors, a contradiction. Name other buyers arbitrarily. Consider first the orientation $\rightarrow$ that is induced by the precedence relation $\prec$ such that

$$3 \prec 1 \prec 2 \prec i \text{ for any } i \geq 4.$$

We then have

$$\begin{aligned}&(s_1, s_2, s_3) = (1, 1, 0)\,,\\ &(d_1 - s_1, d_2 - s_2, d_3 - s_3) = \left(\bar{d} - 1, d_2 - 1, d_3\right).\end{aligned} \tag{52}$$

If s is not a permutation of $d-s$, then we are done. Suppose then that s is a permutation of $d-s$, and consider an alternative orientation $\to'$ of G that is induced by $\prec'$ which agrees with $\prec$ defined above everywhere except between 1 and 2:

$$3 \prec' 2 \prec' 1 \prec' i \text{ for every } i \geq 4, \text{ and } (i \prec' j \Leftrightarrow i \prec j) \text{ for } i,\, j \geq 4.$$

Let $s' = (s_i^{\prec'})_{i \in I}$ be the sequence of in-degrees corresponding to this alternative orientation $\to'$. Then

$$\begin{aligned} &\left(s_1', s_2', s_3'\right) = (0,2,0)\,, \\ &\left(d_1 - s_1', d_2 - s_2', d_3 - s_3'\right) = \left(d_2, \bar{d} - 2, d_3\right). \end{aligned} \tag{53}$$

We also have

$$\begin{aligned} &\left|\left\{i \geq 4 :\, d_i - s_i = 0\right\}\right| = \left|\left\{i \geq 4 :\, d_i - s_i' = 0\right\}\right|, \\ &\left|\left\{i \geq 4 :\, s_i = 0\right\}\right| = \left|\left\{i \geq 4 :\, s_i' = 0\right\}\right|. \end{aligned} \tag{54}$$

The following two cases are considered separately.

a) $d_2 = 1$.

In this case, (52) and (53) imply that

$$\left|\left\{i \leq 3 :\, d_i - s_i = 0\right\}\right| = \left|\left\{i \leq 3 :\, s_i = 0\right\}\right| = 1.$$

Hence, since $d-s$ is a permutation of s, we must have

$$\left|\{i \geq 4 :\, d_i - s_i = 0\}\right| = \left|\{i \geq 4 :\, s_i = 0\}\right|.$$

It then follows from (54) that

$$\left|\left\{i \geq 4 :\, d_i - s_i' = 0\right\}\right| = \left|\left\{i \geq 4 :\, s_i' = 0\right\}\right|. \tag{55}$$

However,

$$\left|\left\{i \leq 3 :\, d_i - s_i' = 0\right\}\right| \leq 1 < 2 = \left|\left\{i \leq 3 :\, s_i' = 0\right\}\right|. \tag{56}$$

(55) and (56) together show that $d-s'$ cannot be a permutation of s'.

b) $d_2 \geq 2$.

In this case, we have $\bar{d} \geq 3$ since $\bar{d} > d_j \geq 2$, and also

$$\left|\left\{i \leq 3 :\, d_i - s_i = 0\right\}\right| = 0 < 1 = \left|\left\{i \leq 3 :\, s_i = 0\right\}\right|.$$

Hence, since $d-s$ is a permutation of s,

$$\left|\{i \geq 4 :\, d_i - s_i = 0\}\right| = \left|\{i \geq 4 :\, s_i = 0\}\right| + 1.$$

It then follows from (54) that

$$\left|\{i \geq 4 : d_i - s'_i = 0\}\right| = \left|\{i \geq 4 : s'_i = 0\}\right| + 1. \tag{57}$$

However, (52) and (53) imply that

$$\left|\left\{i \leq 3 : d_i - s'_i = 0\right\}\right| = \left|\left\{i \leq 3 : s'_i = 0\right\}\right| - 2 \tag{58}$$

(57) and (58) together imply that $d - s'$ is not a permutation of s'.

ii) G is r-regular with $2 < r < N - 1$.

Since G is connected and not complete, we may suppose, with renaming if necessary, that buyers 1 and 2 are adjacent, and that buyer 3 is adjacent to 2 but not to 1. To see that this is possible, suppose to the contrary that for any pair of adjacent buyers i and j, any buyer $k \neq i$ adjacent to j is also adjacent to i. We then show that G must be complete. Take any pair of buyers i and j. Since G is connected, there is a path $k_1 = i \to k_2 \to \cdots \to k_{m-1} \to k_m = j$. Since k_2 is adjacent to $i = k_1$ and k_3 is adjacent to k_2, k_3 is adjacent to i as well by the above. Now since k_4 is adjacent to k_3, it is also adjacent to i. Proceeding the same way, we conclude that $j = k_m$ is adjacent to $i = k_1$, implying that G is complete.

We now name buyers other than 1, 2, and 3 arbitrarily. For the orientation $\to$ induced by $\prec$ such that

$$1 \prec 2 \prec 3 \prec i \text{ for any } i \geq 4,$$

the associated sequence $s = (s_i^{\prec})_{i \in I}$ of indegrees satisfies

$$(s_1, s_2, s_3) = (0, 1, 1)\,,$$
$$(d_1 - s_1, d_2 - s_2, d_3 - s_3) = (r, r - 1, r - 1)\,.$$

If $d - s$ is a not permutation of s, then we are done. Suppose then that $d - s$ is a permutation of s. We then must have

$$\left|\{i : s_i = 0\}\right| = \left|\{i : d_i - s_i = 0\}\right|. \tag{59}$$

Consider an alternative orientation induced by $\prec'$ that agrees with $\prec$ everywhere except between 2 and 3:

$$1 \prec' 3 \prec' 2 \prec' i \text{ for } i \geq 4, \text{ and } i \prec' j \Leftrightarrow i \prec j \text{ for } i, j \geq 4.$$

If we denote by $s' = (s_i^{\prec'})_{i \in I}$ the sequence of indegrees associated with $\prec'$, then it satisfies

$$\left(s'_1, s'_2, s'_3\right) = (0, 2, 0)\,,$$
$$\left(d_1 - s'_1, d_2 - s'_2, d_3 - s'_3\right) = (r, r - 2, r)\,.$$

Since $r > 2$, if (59) holds, then the same argument as in the non-regular case shows that

$$\Big|\{i : s'_i = 0\}\Big| \neq \Big|\{i : d_i - s'_i = 0\}\Big|,$$

implying that $d - s'$ is not a permutation of s'.

2) Suppose that G is cyclic, and let the orientation $\rightarrow \in O_G$ be given. Denote by $s = (s_i^{\prec})_{i \in I}$ the sequence of indegrees associated with $\prec$. Let $J_2 = \{i : s_i = 2\}$ and $J_0 = \{i : s_i = 0\}$, and consider any distinct (and non-adjacent) buyers in J_2 such that there is no buyer in J_2 on the shorter path between them. Clearly, there exists exactly one buyer on that path who belongs to J_0. It follows that $|J_2| = |J_0|$. Since $s_i = d_i - s_i = 1$ for any buyer who is not in J_2 or J_0, s is a permutation of $d - s$.

Suppose next that G is complete, and let the orientation $\rightarrow \in O_G$ be given. If we denote by $s = (s_i^{\prec})_{i \in I}$ the sequence of indegrees associated with $\prec$, then s is a permutation of the sequence $0, 1, 2, \ldots, N-1$. Since $d_i - s_i = (N-1) - s_i$, it follows that $d - s$ is also a permutation of $0, 1, 2, \ldots, N-1$. Hence, s is a permutation of $d - s$. □

Proof of Proposition 8.2. Suppose that G is neither cyclic or complete, and let (p^*, q^*, σ) be any SPE. By Lemma 8.1, $\sum_i (v^{s_i} - v^{d_i - s_i}) > 0$ for some $\rightarrow \in O_G$. Lemma 6.3 then implies that for any such $\rightarrow \in O_G$,

$$\pi_A(p^*, q^*, \sigma) > \sum_{i \in I} \Big(\min\{q_i^*,\, v^{d_i - s_i}\} - c\Big) \geq \sum_{i \in I} \left(\min\{q_i^*,\, v^0\} - c\right). \tag{60}$$

Suppose now that firm B monopolizes the market: $\sigma(p^*, q^*) = (B, \ldots, B)$ so that $\pi_A(p^*, q^*, \sigma) = 0$ and $\pi_B(p^*, q^*, \sigma) = \sum_i (q_i^* - c) \geq 0$. If $\min_i q_i^* \geq c$, then $\pi_A(p^*, q^*, \sigma) = 0 \leq \sum_i \big(\min\{q_i^*, v^0\} - c\big)$, contradicting (60). If $\max_i q_i^* \leq v^0$, then we have a contradiction to (60) since

$$\pi_A(p^*, q^*, \sigma) \leq \pi_B(p^*, q^*, \sigma) = \sum_i (q_i^* - c) = \sum_i \big(\min\{q_i^*, v^0\} - c\big).$$

Note now that $q_i^* \leq v^{d_i} - v^0 + c$ for every i since otherwise firm A can profitably switch buyer i to A by p such that $c < p_i < v^0 - v^{d_i} + q_i^*$ and $p_j = c$ for $j \neq i$. If $v^{\bar{d}} \leq 2v^0 - c$, then we have a contradiction to the above since $\max_i q_i^* \leq v^{\bar{d}} - v^0 + c \leq v^0$. □

Proof of Proposition 8.3 Let $p^* = q^* = (c, \ldots, c)$, and suppose that σ is extremal with respect to (p^*, q^*) with $\sigma(p^*, q^*) = (B, \ldots, B)$.

1) G is a cycle.

Fix any deviation $p \neq p^*$ by firm A and write $D_A^k = D_A^k(p, q^*)$. Since $v^0 - q_i^* = v^0 - c \geq 0$, $x_i = B$ is never dominated by $x_i = \emptyset$ for any buyer i. Furthermore, if $x_i = A$ dominates $x_i = B$, $x_i = A$ also dominates $x_i = \emptyset$. It follows that for any i

and k, either $x_i = A$ is the unique $(k-1)$-rationalizable action ($\{A\} = S_i^{k-1}$), or $x_i = B$ is $(k-1)$-rationalizable ($B \in S_i^{k-1}$). Hence for any i and k,

$$\beta_i^k = 2 - \alpha_i^k.$$

Suppose now that $x_i = A$ is k-dominant ($i \in D_A^k$). Then we have by (9),

$$p_i < \min\left\{v^{\alpha_i^k} - v^{\beta_i^k} + c,\, v^{\alpha_i^k}\right\} = v^{\alpha_i^k} - v^{2-\alpha_i^k} + c.$$

More specifically, i finds $x_i = A$ 1-dominant if $p_i < v^0 - v^2 + c$, and k-dominant for $k > 1$ either if (i) $p_i < c$ and exactly one of i's two neighbors precedes i ($\alpha_i^k = 1$), or (ii) $p_i < v^2 - v^0 + c$ and both his neighbors precede him ($\alpha_i^k = 2$). In particular, if buyer i precedes both his neighbors, then $i \in D_A^1$. Firm A's payoff under (p, q^*, σ) hence satisfies

$$\begin{aligned}\pi_A(p, q^*, \sigma) &= \sum_{k=1}^{K} \sum_{i \in D_A^k} (p_i - c) \\ &< |D_A^1|(v^0 - v^2) + (v^2 - v^0) \sum_{k=2}^{K} \left|\{i \in D_A^k : \alpha_i^k = 2\}\right|.\end{aligned}$$

Since no buyer finds $x_i = A$ k-dominant for $k \geq 2$ if neither of his neighbors precedes him, the number of components (*i.e*, connected clusters of buyers) in $\cup_{\ell=1}^{k-1} D_A^\ell$ is less than or equal to that in D_A^1 for any k.[36] It follows that

$$\sum_{k=2}^{K} |\{i \in D_A^k : \alpha_i^k = 2\}| \leq |D_A^1|.$$

We can therefore conclude that $\pi_A(p, q^*, \sigma) < 0$.

2) G is complete.

Define $D_A^k = D_A^k(p, q^*)$ $(k = 1, \ldots, K)$ as above. Denote by α^k the number of buyers who find $x_i = A$ k-dominant for $1, \ldots, k-1$:

$$\alpha^k = \sum_{\ell=1}^{k-1} \left|D_A^\ell\right|.$$

Since G is complete, for any buyer i, the number α_i^k of i's neighbors who precede him equals α^k. Furthermore, by Lemma 6.2, we only need consider p such that each D_A^k contains a single buyer. (If D_A^k contains two or more buyers, then since G is

[36] A set $J \subset I$ is *connected* if for any pair of buyers i, j in J, there exists a path within J between i and j. For any $k = 1$ or 2, $J \subset I_k$ is a *component* of I_k if it is connected and no $L \subset I_k$ with $L \supsetneq J$ is connected.

complete, those buyers are adjacent.) Hence, without loss of generality, let $D_A^k = \{k\}$ for each $k = 1, \ldots, N$. Just as in the case of a cyclic network, $q^* = (c, \ldots, c)$ implies that $\beta_i^k = d_i - \alpha_i^k$ for any i and k. It follows that

$$p_k < v^{\alpha^k} - v^{N-1-\alpha^k} + c \text{ for } k = 1, \ldots, K.$$

Firm A's payoff under (p, q^*, σ) hence satisfies

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^{N} (p_k - c) < \sum_{k=1}^{K} \left(v^{\alpha^k} - v^{N-1-\alpha^k} \right). \tag{61}$$

It can then readily be verified that the right-hand side is maximized when $K = N$ and that the maximum value equals zero. Hence, firm A has no profitable deviation. □

Proof of Proposition 9.1. If G is either cyclic or complete, then $(p^v, q^v) = (z, z)$ is consistent with a monopolization equilibrium for any externalities v by Proposition 8.3. Conversely, suppose that G is neither cyclic nor complete. Note that if (p^v, q^v) is sufficiently close to (z, z), then $\max_i q_i^v < v^0$ must hold since $v^0 > c$. Proposition 8.2 then shows that (p^v, q^v) cannot be consistent with a monopolization equilibrium unless the externalities are non-generic. □

Proof of Proposition 9.2 The proof uses Lemmas A.5 and A.6 presented below. Given the externalities v, we begin by defining the price vector z^v. As v approaches h-linear externalities, this price vector z^v approaches the bipartition price vector z defined in (19). Fix any orientation $\rightarrow \in O_G^*$ and let $s = (s_i^{\rightarrow})_{i \in I}$. Let

$$\zeta_1 = \sum_{i \in I_1} \left(v^{d_i} - v^0 \right). \tag{62}$$

Suppose that I_2 consists of T components $(I_{2t})_{t=1,\ldots,T}$.[37] For each component I_{2t}, let

$$\zeta_{2t} = \max_{\rightarrow \in O_G^*} \sum_{i \in I_{2t}} \left(v^{s_i} - v^{d_i - s_i} \right), \quad \text{and} \quad \zeta_2 = \sum_{t=1}^{T} \zeta_{2t}. \tag{63}$$

Define the set V^* of externalities by

$$V^* = \{ v : \Delta \equiv \zeta_1 - \zeta_2 \geq 0 \}.$$

V^* is non-degenerate as will be seen below, and contains the h-linear externalities as one of its elements since then $\zeta_1 = \zeta_2$. Let now a price vector z^v be defined by

$$z_i^v = \begin{cases} v^{d_i^2} - v^0 + c & \text{if } i \in I_1, \\ -\frac{\zeta_{2t}}{\sum_{j \in I_{2t}} d_j^1} d_i^1 + c. & \text{if } i \in I_{2t}. \end{cases} \tag{64}$$

[37]See Figure 5 and Footnote 36.

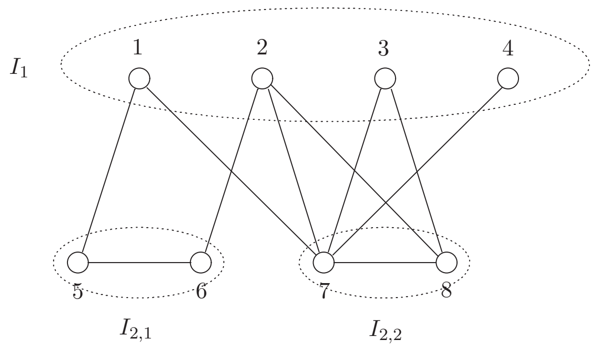

Figure 5. Components of I_2: $I_2 = I_{2,1} \cup I_{2,2}$.

In Figure 5, for example, $\zeta_{2,1}$ and $\zeta_{2,2}$ equal

$$\zeta_{2,1} = v^2 - v^0 \quad \text{and} \quad \zeta_{2,2} = \max\left\{v^4 - v^1 + v^3 - v^0,\ v^2 - v^1 + v^5 - v^0\right\},$$

and hence the price vector z is given by

$$\begin{aligned} &z_1^v = z_3^v = v^2 - v^0 + c, \quad z_2^v = v^3 - v^0 + c, \quad z_4^v = v^1 - v^0 + c, \\ &z_5^v = z_6^v = -\frac{1}{2}\,\zeta_{2,1} + c, \quad z_7^v = -\frac{1}{2}\,\zeta_{2,2} + c \quad \text{and} \quad z_8^v = -\frac{2}{3}\,\zeta_{2,2} + c. \end{aligned}$$

Note that as v approaches h-linear externalities, $v^{d_i^2} - v^0 \to d_i^2 h$, and $\frac{\zeta_{2t}}{\sum_{j \in I_{2t}} d_j^1} \to h$ so that $z^v \to z$ for the bipartition price vector z defined in (19).

In what follows, we set $p^* = q^* = z^v$ and let σ be extremal with respect to (p^*, q^*) with monopolization by firm B on the path: $\sigma(p^*, q^*) = (B, \ldots, B)$. By construction, we have

$$\pi_A(p^*, q^*, \sigma) = 0 \quad \text{and} \quad \pi_B(p^*, q^*, \sigma) = \sum_{i \in I} (z_i^v - c) = \zeta_1 - \zeta_2 \geq 0.$$

We first note that the set V^* is non-degenerate under the stated condition of Proposition 9.2. For this, define

$$\varepsilon_d = v^d - v^0 - dh \text{ for } d = 1, \ldots, \bar{d}, \text{ and } \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{\bar{d}}).$$

Note that

$$v \in V^* \quad \Leftrightarrow \quad \lambda^{\to} \cdot \varepsilon \equiv \lambda_1^{\to} \varepsilon_1 + \cdots + \lambda_{\bar{d}}^{\to} \varepsilon_{\bar{d}} \leq 0 \text{ for every } \to \in O_G^*.$$

To show that V^* is non-degenerate, hence, it suffices to show that the set defined by the corresponding strict inequalities is non-empty:

$$\left\{\varepsilon \in \boldsymbol{R}^{\bar{d}} : \lambda^{\to} \cdot \varepsilon < 0 \text{ for every } \to \in O_G^* \text{ such that } \lambda^{\to} \neq 0\right\} \neq \emptyset.$$

By Theorem 22.2 of Rockafellar (1997, p198), this holds if and only if no convex combination of the collection of vectors $\{\lambda^{\rightarrow} : \lambda^{\rightarrow} \neq 0, \rightarrow \in O_G^*\}$ equals zero, as required by the proposition.

The remainder of the proof shows that no deviation p by firm A is profitable when $v \in Z^*$ is ε-close to h-linear, z^v satisfies $|z_i^v - z_i| < \varepsilon$ for every i, and $\varepsilon < \min\{\frac{h}{3N}, \frac{v^0-c}{2}, \frac{h-v^0+c}{2d}\}$. By Lemma A.6, we may restrict attention to p for which $\beta_i^k = d_i - \alpha_i^k$ for any i and k. In this case, if $x_i = A$ is k-dominant for buyer i (*i.e.*, $i \in D_A^k$), then it should satisfy

$$p_i < \min\left\{v^{\alpha_i^k} - v^{\beta_i^k} + q_i^*, \, v^{\alpha_i^k}\right\} \leq v^{\alpha_i^k} - v^{d_i-\alpha_i^k} + q_i^*. \tag{65}$$

We now proceed in the following steps.

1. Any profitable deviation p attracts at least one buyer in I_1: $I_1 \cap D_A \neq \emptyset$.

   If $D_A \subset I_2$, then every neighbor of $i \in D_A$ should also belong to D_A since otherwise, $L(I_2 \cap D_A, I \setminus D_A) \geq 1$ and hence $\pi_A(p, q^*, \sigma) < 0$ under approximate linearity by Lemma A.5. Proceeding iteratively, every buyer who is connected to i should belong to D_A. Since the network G is connected, there must exist $j \in I_1 \cap D_A$, a contradiction.

2. Any profitable deviation p attracts at least one buyer in I_2 and all his neighbors: $I_2 \cap D_A \neq \emptyset$ and $N_i \subset D_A$ for $i \in I_2 \cap D_A$.

   Since I_1 is independent, if $D_A \subset I_1$, then

$$\begin{aligned}\pi_A(p, q^*, \sigma^B) &< \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i-\alpha_i^k}\right) + \sum_{i \in D_A} (q_i^* - c) && \Leftarrow (65)\\ &= \sum_{i \in D_A} \left(v^0 - v^{d_i}\right) + \sum_{i \in D_A} \left(v^{d_i} - v^0\right) && \Leftarrow (62)\\ &= 0.\end{aligned}$$

   It follows that if p is a profitable deviation, then $I_2 \cap D_A \neq \emptyset$. If $N_i \setminus D_A \neq \emptyset$ for some $i \in I_2 \cap D_A$, there is a link between $I_2 \cap D_A$ and $I \setminus D_A$. Since $\beta_i^k = d_i - \alpha_i^k$ for every i and k, $\pi_A(p, q^*, \sigma) < 0$ by Lemma A.5.

3. If p is any profitable deviation, then $x_i = A$ is 1-dominant for any buyer $i \in I_1$ and is k-dominant for $k \geq 2$ for any buyer $i \in I_2$: $D_A^1 \subset I_1$, and $D_A^k \subset I_2$ for $k \geq 2$.

   i) If $i \in I_2$, then $S_i^k \subset \{A, B\}$ for $k = 1, \ldots, K$.
   If $i \in I_2$, $v^0 - q_i^* = v^0 + d_i^1 h - c > 0$ so that $x_i = \emptyset$ is dominated by $x_i = B$ in S^0.

ii) If $i \in I_1 \cap D_A^k$ for some $k \geq 2$, then $N_i \cap D_A^{k-1} \neq \emptyset$.
Suppose to the contrary that $i \in I_1 \cap D_A^k$ and $N_i \cap D_A^{k-1} = \emptyset$ for some i and $k \geq 2$. We then have $\alpha_i^{k-1} = \alpha_i^k$. Furthermore, since $S_j^{k-1} \subset \{A, B\}$ for every $j \in N_i \subset I_2$ by (3i), $j \notin D_A^{k-1}$ implies that $B \in S_j^{k-2} \Rightarrow B \in S_j^{k-1}$. It follows that $\beta_i^{k-1} = \beta_i^k$. Hence, if $x_i = A$ is not $(k-1)$-dominant, then it cannot be k-dominant since

$$v^{\alpha_i^{k-1}} - p_i \leq \max\left\{v^{\beta_i^{k-1}} - q_i^*, 0\right\} \quad \Rightarrow \quad v^{\alpha_i^k} - p_i \leq \max\left\{v^{\beta_i^k} - q_i^*, 0\right\}.$$

This is a contradiction to $i \in D_A^k$.

iii) If $I_1 \cap D_A^k \neq \emptyset$ for some $k \geq 2$, then p is not profitable.
Let $k \geq 2$ and $i \in I_1 \cap D_A^k$. We have $N_i \cap D_A^{k-1} \neq \emptyset$ by (3ii). It then follows that $\beta_i^k \leq |N_i| - 1 = d_i^2 - 1$. By Lemma A.6, we conclude that $\pi_A(p, q^*, \sigma^B) < 0$.

iv) If $i \in I_2 \cap D_A^1$, then p is not profitable.
Let $i \in I_2 \cap D_A^1$. Since I_1 is maximally independent, there exists $j \in N_i \cap I_1$. For this j, we have $\beta_j^2 \leq |N_j| - 1 = d_j^2 - 1$ and hence $\pi_A(p, q^*, \sigma^B) < 0$ by Lemma A.6.

v) If p is a profitable deviation, then $D_A^1 \subset I_1$, $D_A^k \subset I_2$ for $k \geq 2$, $N_i \subset D_A$ for $i \in I_2 \cap D_A$.
Suppose that $\pi_A(p, q^*, \sigma) \geq 0$. By (3iii) and (3iv), we must have $D_A^1 \subset I_1$ and $D_A^k \subset I_2$ for $k \geq 2$. By Step 2, $N_i \subset D_A$ for $i \in I_2 \cap D_A$. □

4. No deviation p is profitable.

By Step 3, we may restrict attention to p such that $D_A^1(p, q^*) \subset I_1$, and $D_A^k(p, q^*) \subset I_2$ for $k \geq 2$. By Steps 1 and 2, we may also suppose that $I_1 \cap D_A \neq \emptyset$ and that $I_2 \cap D_A$ consists of components of I_2 that are adjacent only to $I_1 \cap D_A$. Suppose for simplicity that $I_2 \cap D_A$ consists of a single component I_{2t} of I_2. Then

$$\begin{aligned}
&\pi_A(p, q^*, \sigma^B) \\
&< \sum_{i \in D_A^1} (v^0 - v^{d_i^2} + q_i^* - c) + \sum_{k=2}^{K} \sum_{i \in D_A^k} (v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* - c) \qquad \Leftarrow (65) \\
&= \sum_{k=2}^{K} \sum_{i \in D_A^k} (v^{\alpha_i^k} - v^{d_i - \alpha_i^k}) - \zeta_{2t} \qquad \Leftarrow (62) \text{ and } (63) \\
&\leq 0.
\end{aligned}$$

□

Lemma A.5 *Let the network G and the bipartition (I_1, I_2) of the buyer set be given. For each $\varepsilon > 0$, take externalities $v(\varepsilon)$ and the price vector $q^*(\varepsilon)$ such that $v(\varepsilon)$ is ε-close to h-linear ($h > 0$), and that $|q_i^*(\varepsilon) - z_i| < \varepsilon$ for every i for the bipartition price vector z given (I_1, I_2). Then for $\varepsilon > 0$ sufficiently small, $\pi_A(p, q^*(\varepsilon), \sigma^B) < 0$ for any p such that $\beta_i^k = d_i - \alpha_i^k$ for every i and k, and $L(D_A, I_1 \backslash D_A) + L(I_2 \cap D_A, I \backslash D_A) \geq 1$.*

Proof. Let $\varepsilon < \frac{h}{3N}$. Take p such that $\beta_i^k = d_i - \alpha_i^k$ for every i and k. Then

$$\begin{aligned}
\pi_A(p, q^*(\varepsilon), \sigma^B) &= \sum_{k=1}^{K} \sum_{i \in D_A^k} (p_i - c) \\
&< \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(\min \left\{ v^{\alpha_i^k}(\varepsilon) - v^{\beta_i^k}(\varepsilon) + q_i^*(\varepsilon),\ v^{\alpha_i^k}(\varepsilon) \right\} - c \right) \\
&\leq \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k}(\varepsilon) - v^{d_i - \alpha_i^k}(\varepsilon) + q_i^*(\varepsilon) - c \right) \\
&< \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k}(0) - v^{d_i - \alpha_i^k}(0) + q_i^*(0) - c \right) + 3|D_A|\varepsilon \\
&\leq -h + 3|D_A|\varepsilon < 0,
\end{aligned}$$

where next to the last inequality follows from Lemma A.4 and our assumption that $L(D_A, I_1 \setminus D_A) + L(I_2 \cap D_A, I \setminus D_A) \geq 1$, and the last inequality from our choice of ε. □

Lemma A.6 *Let the buyer network G and the bipartition (I_1, I_2) of the buyer set be given. Take h, v^0, and $c \geq 0$ such that $h > v^0 - c > 0$. For each $\varepsilon > 0$, take externalities $v(\varepsilon)$ and price vector $q^*(\varepsilon)$ such that $v(\varepsilon)$ is ε-close to h-linear, and that $|q_i^*(\varepsilon) - z_i| < \varepsilon$ for every i for the bipartition price vector z given (I_1, I_2) defined in (19). Then for ε sufficiently small, if p is such that $\pi_A(p, q^*(\varepsilon), \sigma^B) \geq 0$, then the following holds for every i and k:*

$$\begin{aligned}
&\textit{(i) } x_i = B \textit{ is } k\textit{-rationalizable } (B \in S_i^k), \textit{ or} \\
&\textit{(ii) } x_i = A \textit{ is the unique } k\textit{-rationalizable action } (\{A\} = S_i^k)
\end{aligned} \qquad (66)$$

Furthermore, if p is any such price vector, then $\alpha_i^k + \beta_i^k = d_i$ for every i and k.

Proof. Take $\varepsilon < \min\{\frac{h}{3N}, \frac{v^0-c}{2}, \frac{h-v^0+c}{2d}\}$. First, since $q_i^*(\varepsilon) < z_i + \varepsilon = -d_i^1 h + c + \varepsilon < v^0$ for $i \in I_2$, (66) may fail only for $i \in I_1$. For $i \in I_1$, $\varepsilon < \frac{v^0-c}{2}$ implies that

$$\begin{aligned} v^{\beta_i^k}(\varepsilon) < q_i^*(\varepsilon) \Rightarrow \quad & v^0 + h\beta_i^k - \varepsilon < c + hd_i^2 + \varepsilon \\ \Leftrightarrow \quad & d_i^2 - \beta_i^k > \frac{v^0 - c - 2\varepsilon}{h} \\ \Rightarrow \quad & d_i^2 - \beta_i^k \geq 1. \end{aligned} \tag{67}$$

Under the simplifying assumption that $i \in I_1$ is the only buyer for whom (66) fails, the same logic as in the proof of Lemma A.2 leads to the following inequality which is the same as (43):

$$\begin{aligned} \pi_A(p, q^*, \sigma^B) &< \sum_{\ell=1}^{K} \sum_{j \in D_A^\ell} \left(v^{\alpha_j^\ell}(\varepsilon) - v^{d_j - \alpha_j^\ell}(\varepsilon) + q_j^*(\varepsilon) - c \right) \\ &\quad + \sum_{\ell=k+1}^{m-1} \sum_{j \in N_i \cap D_A^\ell} \left(v^{d_j - \alpha_j^\ell}(\varepsilon) - v^{d_j - \alpha_j^\ell - 1}(\varepsilon) \right) + v^{d_i - \alpha_i^m}(\varepsilon) - q_i^*(\varepsilon) \\ &\leq \sum_{\ell=k+1}^{m-1} \sum_{j \in N_i \cap D_A^\ell} \left(v^{d_j - \alpha_j^\ell}(\varepsilon) - v^{d_j - \alpha_j^\ell - 1}(\varepsilon) \right) + v^{d_i - \alpha_i^m}(\varepsilon) - q_i^*(\varepsilon), \end{aligned} \tag{68}$$

where this time, the second inequality follows from Lemma A.5 as $\varepsilon < \frac{h}{3N}$. Since

$$v^{d_j - \alpha_j^\ell}(\varepsilon) - v^{d_j - \alpha_j^\ell - 1}(\varepsilon) < h + 2\varepsilon,$$

and

$$v^{d_i - \alpha_i^m}(\varepsilon) - q_i^*(\varepsilon) < v^0 - c - h(d_i^2 - d_i + \alpha_i^m) + 2\varepsilon,$$

(68) implies

$$\pi_A(p, q^*(\varepsilon), \sigma^B) < \sum_{\ell=k+1}^{m-1} \sum_{j \in N_i \cap D_A^\ell} (h + 2\varepsilon) + v^0 - c - h(d_i^2 - d_i + \alpha_i^m) + 2\varepsilon. \tag{69}$$

When $m - 1 \geq k + 1$, (69) reduces to

$$\begin{aligned} \pi_A(p, q^*(\varepsilon), \sigma^B) &< (\alpha_i^m - \alpha_i^{k+1})(h + 2\varepsilon) + v^0 - c - h(d_i^2 - d_i + \alpha_i^m) + 2\varepsilon \\ &= h(d_i - \alpha_i^{k+1} - d_i^2) + v^0 - c + 2\varepsilon(\alpha_i^m - \alpha_i^{k+1} + 1) \\ &= h(\beta_i^{k+1} - d_i^2) + v^0 - c + 2\varepsilon(\alpha_i^m - \alpha_i^{k+1} + 1) \\ &\leq h(\beta_i^k - d_i^2) + v^0 - c + 2d_i\varepsilon \\ &\leq -h + v^0 - c + 2d_i\varepsilon. \end{aligned}$$

By the same logic as in the proof of Lemma A.2, the same inequality holds true when $m = k + 1$ or $m = k$, or when $i \notin D_A$. Since $-h + v^0 - c + 2d_i\varepsilon < 0$ by our choice of ε, $\pi_A(p, q^*(\varepsilon), \sigma^B) < 0$ for any p that fails (66). □

Proof of Corollary 9.3

1) Let $e = (1, \ldots, 1) \in \boldsymbol{R}^{\bar{d}}$. Then $e \cdot \lambda \leq |J'| - |I_1|$ for any $\lambda \in \Lambda$, where J' is the set that has the largest number of elements among independent sets which are subsets of I_2. By assumption, $|J'| - |I_1| < 0$ so that $e \cdot \lambda < 0$ for any $\lambda \in \Lambda$. It follows that for any convex combination $\sum_k \zeta_k \lambda_k$ of vectors $(\lambda_k)_k$ in Λ, we have $e \cdot \sum_k \zeta_k \lambda_k = \sum_k \zeta_k (e \cdot \lambda_k) < 0$ so that $\sum_k \zeta_k \lambda_k \neq 0$.

2) Let $\hat{e} = (0, \ldots, 0, 1) \in \boldsymbol{R}^{\bar{d}}$. Then $\hat{e} \cdot \lambda = -|\{i \in I_1 : d_i = \bar{d}\}| < 0$ for any $\lambda \in \Lambda$. It then follows from the same argument as above that no convex combination of vectors in Λ equals zero.

3) When every component of I_2 is a singleton, every orientation $\rightarrow \in O_G^*$ yields the same vector $\lambda^{\rightarrow} = \lambda$ so that Λ is a singleton. Hence, the condition is satisfied. □

Proof of Proposition 10.1 Since I_2 is independent, for each $i \in I_2$, $\{i\}$ is a component of I_2. Hence, ζ_{2t} in (63) in the proof of Proposition 9.2 equals

$$\zeta_{2t} = v^{d_t^1} - v^0 \quad \text{for each } t \in I_2,$$

implying that the price vector $p^* = q^*$ given in the proposition equals z^v in (64). Furthermore, the set V^* of externalities defined in the proof of Proposition 9.2 is given by $V^* = \{v : \sum_{i \in I_1} (v^{d_i^2} - v^0) \geq \sum_{i \in I_2} (v^{d_i^1} - v^0)\}$. Proposition 9.2 then shows that for $\varepsilon > 0$ sufficiently small, if $v \in V^*$ is ε-close to h-linear, then there exists an equilibrium with $p^* = q^* = z^v$. □

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Supplementary Material

(Not to be included in the paper.)

B Equilibrium with Market Segmentation

This section analyzes an equilibrium with market segmentation by the two firms under linearity. We say that an equilibrium (p^*, q^*, σ) entails *complete market segmentation* if every buyer chooses either A or B: *i.e.,*there exists a bipartition (I_1, I_2) of the buyer set I such that $I_1,\ I_2 \neq \emptyset$ and $I_1 = \{i : \sigma_i(p^*, q^*) = A\}$ and $I_2 = \{i : \sigma_i(p^*, q^*) = B\}$.

The following proposition presents a sufficient condition for the non-existence of an equilibrium with complete market segmentation. As can be readily verified, this condition holds in the line network of Section 4.

Proposition B.1 *(Non-existence of a complete segmentation equilibrium) Suppose that the buyer network G is such that for any bipartition (I_1, I_2) of the set of buyers,*

$$\sum_{i \in I_1} \left(d_i^1 - d_i^2\right) < 0, \quad or \quad \sum_{i \in I_2} \left(d_i^2 - d_i^1\right) < 0. \tag{70}$$

Suppose that the externalities are h-linear for $h > 0$. Then there exists no equilibrium with complete market segmentation.

Proof. Suppose that for some bipartition (I_1, I_2) with $I_1,\ I_2 \neq \emptyset$, there exists an equilibrium (p^*, q^*, σ) such that buyers in I_1 choose firm A, and buyers in I_2 choose firm B in equilibrium. For any $j \in I_1$, $v^{d_j^1} - p_j^* \geq \max\left\{v^{d_j^2} - q_j^*,\ 0\right\}$ so that $p_j^* \leq v^{d_j^1}$ and $q_j^* \geq v^{d_j^2} - v^{d_j^1} + p_j^* = (d_j^2 - d_j^1)h + p_j^*$. Suppose without loss of generality that

$$\sum_{j \in I_1} \left(d_j^1 - d_j^2\right) < 0. \tag{71}$$

Suppose now that firm A offers a DC price vector p such that under (p, q^*), buyers in I_1 precede those in I_2 in the elimination process. Since buyer i finds A dominant if $p_i < \min\left\{v^{s_i} - v^{d_i - s_i} + q_i^*,\ v^{s_i}\right\}$, firm A's payoff from buyers in I_1 is bounded from below by

$$\begin{aligned}
-L(I_1, I_2)h + \sum_{j \in I_1} (q_j^* - c) &\geq -hL(I_1, I_2) + \sum_{j \in I_1} \left\{p_j^* - c + (d_j^2 - d_j^1)h\right\} \\
&= -hL(I_1, I_2) + \sum_{j \in I_1} (p_j^* - c) + h \sum_{j \in I_1} (d_j^2 - d_j^1).
\end{aligned}$$

On the other hand, by setting $K = I_1$ and $J = I_2$ in Lemma A.3 and noting that $\bar{q}_j^* = \min\{q_j^*, v^{d_j^2}\} = q_j^*$ for $j \in I_2$, we see that firm A's payoff from buyers in I_2 is bounded from below by

$$r(I_2 \mid I_1, q^*) \geq hL(I_1, I_2) + \min\Big\{\sum_{j \in I_2} (q_j^* - c),\ |I_2|(v^0 - c) + hL(I_2, I_1)\Big\}.$$

Since (p^*, q^*, σ) is an equilibrium, $\sum_{j \in I_2}(q_j^* - c) \geq 0$ so that $r(I_2 \mid I_1, q^*) \geq hL(I_1, I_2)$. It follows that firm A's payoff from offering a DC price vector p satisfies

$$\pi_A(p, q^*, \sigma) \geq \sum_{j \in I_1} (p_j^* - c) + h \sum_{j \in I_1} (d_j^2 - d_j^1) > \sum_{j \in I_1} (p_j^* - c) = \pi_A(p^*, q^*, \sigma),$$

which is a contradiction to our assumption that (p^*, q^*, σ) is an equilibrium. □

It remains an open question whether the existence of a bipartition (I_1, I_2) such that $\sum_{i \in I_1}(d_i^1 - d_i^2) \geq 0$ and $\sum_{i \in I_2}(d_i^2 - d_i^1) \geq 0$ ensures the existence of a complete segmentation equilibrium. It is, however, not difficult to show that MC cost pricing is consistent with a complete segmentation equilibrium under h-linearity if there exists a bipartition (I_1, I_2) such that $d_i^1 \geq d_i^2$ for every $i \in I_1$ and $d_i^2 \geq d_i^1$ for every $i \in I_2$. In other words, (p, q, σ) is a complete segmentation equilibrium if $(p, q) = (z, z)$ for $z = (c, \ldots, c)$ and σ is extremal with respect to (p, q) with $\sigma_i(p, q) = A$ for $i \in I_1$ and $\sigma_i(p, q) = B$ for $i \in I_2$.

In what follows we show a slightly stronger condition on the network guarantees the existence of a complete segmentation equilibrium when the externalities are approximately linear. Specifically, we say that the buyer network is *bi-cohesive* if there exists a bipartition (I_1, I_2) of the set I of buyers such that for m, $n = 1$, 2, and $m \neq n$,

$$|N_i \cap I_n| \geq |N_i \cap I_m| \text{ for every } i \in I_n, \text{ and}$$
$$|N_i \cap I_n| > |N_i \cap I_m| \text{ for some } i \in I_n.$$

Intuitively, in each element of a bipartition (I_1, I_2), there are *core* and *peripheral* buyers: The core buyers are those who have strictly more neighbors in the same set than in the other set, while the peripheral buyers have as many neighbors in the same set as in the other set. The definition of a bi-cohesive network is a natural extension of a *cohesive set* that is studied extensively in social network theory, and is also closely related to the definition of a *cohesive network* proposed by Morris (2000).[38] A line of four or more buyers is bi-cohesive if I_1 consists of at least two buyers on the left, and I_2 consists of at least two buyers on the right. The buyers on the two ends can be taken as core buyers in this case. The regular network in Figure 2 is also bi-cohesive when we take $I_1 = \{1, 2, 3, 4\}$ and $I_2 = \{5, 6, 7, 8\}$. Buyers 2 and

[38] See for example Seidman (1983) for the various definitions of a cohesive set. Morris (2000) defines cohesion in terms of the ratio of neighbors in the same subset over those in the other subset.

3 are core buyers for I_1 and buyers 6 and 7 are core buyers for I_2. More generally, the following lemma describes a sufficient condition for a network to be bi-cohesive based on the notion of connectedness.[39] Two paths from i to j are *disjoint* if they have no link in common. For any integer n, we say that a subset $J \subset I$ is *n-connected* if for any i, $j \in J$, there are n disjoint paths within J connecting i to j.

Lemma B.2 *(Buyer clusters and bi-cohesiveness) A network is bi-cohesive if for an integer n, there exist two n-connected disjoint sets of buyers J_1 and J_2 with $|J_1|$, $|J_2| \geq 2$ such that there exist at most n disjoint paths connecting J_1 to J_2.*

Proof. Let $m \leq n$ be the number of disjoint paths connecting J_1 and J_2. By Menger's theorem, there exists a set L of m links such that removal of those links separates J_1 from J_2. Denote $L = \{i_1 j_1, \ldots, i_m j_m\}$, where $i_k j_k$ is the link between buyers i_k and j_k ($k = 1, \ldots, m$). Define I_1 and I_2 to be the sets of buyers that are connected to J_1 and J_2, respectively, after the removal of links in L. Without loss of generality, we name the links so that $i_1, \ldots, i_m$ are connected to J_1, and $j_1, \ldots, j_m$ are connected to J_2, after the removal of L. Take i_1. If $i_1 \in J_1$, then i_1 has at least n neighbors in $J_1 \subset I_1$ and at most m neighbors in I_2. If $i_1 \notin J_1$, then i_1 again has at least as many neighbors in I_1 as he does in I_2 since otherwise, we would have a smaller set L' of links than L whose removal separates J_1 from J_2. If there is a buyer in I_1 other than $i_1, \ldots, i_m$, then he is a core buyer of I_1 since he has no neighbor in I_2 by definition. If, on the other hand, there is no other buyer in I_1 than $i_1, \ldots, i_m$, then at least one of them is a core buyer: Since $|J_1| \geq 2$, $m \geq 2$ so that at least one of $i_1, \ldots, i_m$ has strictly fewer neighbors in I_2 than m, whereas he has at least $n \geq m$ neighbors in $J_1 \subset I_1$. $\square$

According to Lemma B.2, a network is bi-cohesive if there are at least two "clusters" of buyers who are closely linked among themselves. Such clusters can be a natural consequence of geographic and other proximity among some subsets of buyers.

Proposition B.3 *(Complete segmentation equilibrium under approximate linearity) Suppose that G is bi-cohesive. For any $h > 0$, there exists $\varepsilon > 0$ such that if the externalities are ε-close to h-linear, there exists an SPE (p^*, q^*, σ) in which buyers in I_1 choose firm A and buyers in I_2 choose firm B. In this SPE, $p^*_{i_1} = c + \delta$ and $q^*_{i_1} = c - \delta$ for a single core buyer $i_1 \in I_1$, $p^*_{i_2} = c - \delta$ and $q^*_{i_2} = c + \delta$ for a single core buyer $i_2 \in I_2$, and $p^*_i = q^*_i = c$ for any other buyer i, where*

$$\delta = \max_{\prec \in O_G} \sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} \right). \tag{72}$$

[39] In social network theory, cohesiveness is often defined in terms of connectedness given that an efficient algorithm exists for its computation.

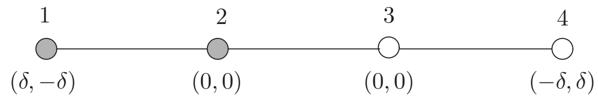

Figure 6. Segmentation equilibrium on a line network ($\delta = |v^2 + v^1 - 2v^0| > 0$): A captures $I_1 = \{1,2\}$ and B captures $I_2 = \{3,4\}$. Indicated next to each buyer is the markup or markdown specified by both firms: $(p_i^* - c, q_i^* - c)$.

As seen, δ is the maximal benchmark payoff that is strictly positive under generic externalities (Lemma 8.1). Each firm's equilibrium payoff equals δ, while the sum of the markups and markdowns over all buyers equals zero (*i.e.*, $\sum_i (p_i^* - c) = \sum_i (q_i^* - c) = 0$). Figure 6 illustrates the equilibrium for the line network of four buyers. As in Proposition 6.1, any deviation by either firm results in the play of the extreme equilibrium that least favors the deviating firm. Note that the core-periphery pricing strategy is a natural form of price discrimination: It charges a markup to a core buyer of the own market segment who finds it more difficult to unilaterally switch to the other firm because of the adoption decision of the majority of his neighbors. To see that neither firm has an incentive to deviate, suppose that firm A employs a DC price vector p. We can verify that $\beta_i^k(p, q^*) = d_i - \alpha_i^k(p, q^*)$ for every i for whom $x_i = A$ is k-dominant. It follows that

$$\begin{aligned}
\pi_A(p, q^*, \sigma^B) &< \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(\min\left\{ v^{\alpha_i^k} - v^{\beta_i^k} + q_i^*,\, v^{\alpha_i^k} \right\} - c \right) \\
&\le \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} \right) \qquad \Leftarrow \sum_i (q_i^* - c) = 0 \\
&\le \delta = \pi_A(p^*, q^*, \sigma).
\end{aligned}$$

As noted above, MC pricing is consistent with a complete segmentation equilibrium in a bi-cohesive network. Since $\delta \to 0$ in (73) as the externalities approach h-linearity, the pricing strategy of Proposition B.3 equals MC pricing in the limit. Hence, we readily conclude that MC pricing under linearity is robust for a segmentation equilibrium in a bi-cohesive network.

Corollary B.4 *(Robustness of MC pricing for a segmentation equilibrium in a bi-cohesive network) Suppose that the network G is bi-cohesive with bipartition (I_1, I_2), and let $(p, q) = (z, z)$ for $z = (c, \ldots, c)$ and σ be extremal with respect to (p, q) with $\sigma_i(p, q) = A$ for $i \in I_1$ and $\sigma_i(p, q) = B$ for $i \in I_2$. Then the equilibrium (p, q, σ) under h-linearity is robust.*

Proof of Proposition B.3. Let

$$\delta = \max_{\prec \in O_G} \sum_{i \in I} \left(v^{s_i} - v^{d_i - s_i} \right).$$

When the externalities are ε-close to h-linear,

$$\begin{aligned}\sum_{i\in I} \left(v^{s_i} - v^{d_i-s_i}\right) &= \sum_{i \in I} \left\{ (v^{s_i} - s_i h) - \left(v^{d_i - s_i} - (d_i - s_i)h\right) - h\left((d_i - s_i) - s_i\right)\right\} \\ &< 2N\varepsilon,\end{aligned}$$

and hence

$$\delta < 2N\varepsilon. \tag{73}$$

Let (I_1, I_2) be the partition of the buyer set I, and let $i_A \in I_1$ and $i_B \in I_2$ be the core buyers of the respective sets:

$$|N_{i_A} \cap I_1| > |N_{i_A} \cap I_2| \text{ and } |N_{i_B} \cap I_2| > |N_{i_B} \cap I_1|.$$

We specify (p^*, q^*, σ) as follows:

$$(p_i^*, q_i^*) = \begin{cases} (\delta + c, -\delta + c) & \text{if } i = i_A, \\ (-\delta + c, \delta + c) & \text{if } i = i_B, \\ (c, c) & \text{otherwise,} \end{cases}$$

and

$$\sigma(p, q) = \begin{cases} (\underbrace{A, \ldots, A}_{I_1}, \underbrace{B, \ldots, B}_{I_2}) & \text{if } (p, q) = (p^*, q^*), \\ \sigma^B(p, q) & \text{if } p \neq p^*, \\ \sigma^A(p, q) & \text{if } p = p^* \text{ and } q \neq q^*. \end{cases}$$

Note that $\pi_A(p^*, q^*, \sigma) = \pi_B(p^*, q^*, \sigma) = \delta$.

We first show that the buyers' action profile following (p^*, q^*) is a NE. If $i \in I_1 \setminus \{i_A\}$, then $x_i = A$ is a best response since

$$v^{|N_i \cap I_1|} - p_i = v^{|N_i \cap I_1|} - c \geq v^{|N_i \cap I_2|} - c = v^{|N_i \cap I_2|} - q_i.$$

If $i = i_A$, then $|N_i \cap I_1| > |N_i \cap I_2|$ so that

$$\begin{aligned}&v^{|N_i \cap I_1|} - v^{|N_i \cap I_2|} \\ &= \left(v^{|N_i \cap I_1|} - h|N_i \cap I_1|\right) - \left(v^{|N_i \cap I_2|} - h|N_i \cap I_2|\right) + h\left\{|N_i \cap I_1| - |N_i \cap I_2|\right\} \\ &\geq h - 2\varepsilon.\end{aligned}$$

Hence, if we take

$$\bar{\varepsilon} = \frac{h}{2(2N+1)}, \tag{74}$$

then for any $\varepsilon < \bar{\varepsilon}$, (73) implies that

$$v^{|N_i \cap I_1|} - p_i = v^{|N_i \cap I_1|} - \delta - c > v^{|N_i \cap I_2|} + \delta - c = v^{|N_i \cap I_2|} - q_i.$$

The symmetric argument shows that $x_i = B$ is a best response for each $i \in I_2$ following (p^*, q^*).

We will next show that firm A has no profitable deviation. Let p be any deviation by firm A, and $D_A^k = D_A^k(p, q^*)$. Suppose that $i \in D_A^k$. By Lemma 6.2, we may assume that $N_i \cap D_A^k = \emptyset$. For any $j \in N_i$, we observe that

$$j \notin \cup_{\ell=1}^{k-1} D_A^\ell \quad \Rightarrow \quad B \in S_j^{k-1}. \tag{75}$$

We can see that (75) holds as follows: First, take $j \neq i_B$. Since then $q_j^* \leq c \leq v^0$, $x_j = B$ is not dominated by $x_j = \emptyset$. Hence, if $x_j = A$ is not dominant in $S^{\ell-1}$ for $\ell = 1, \ldots, k-1$ (*i.e.*, $j \notin \cup_{\ell=1}^{k-1} D_A^\ell$), then $B \in S_j^{k-1}$.

On the other hand, if $j = i_B$, then $q_j = \delta + c$. Since $i \in D_A^k$. $i \notin \cup_{\ell=1}^{k-1} D_A^\ell$. Since $i \neq i_B$, $B \in S_i^{k-1}$ by the above. It follows that $\beta_j^k \geq 1$. Hence,

$$v^{\beta_j^k} - q_j^* \geq v^1 - \delta - c > v^0 + h - \varepsilon - \delta - c > 0.$$

Hence, (75) holds for any i, which in turn implies that $\beta_i^k = d_i - \alpha_i^k$ for any i and k. Hence, by (9), if $i \in D_A^k$, then

$$p_i < \min\left\{v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*,\, v^{\alpha_i^k}\right\} \leq v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*.$$

Therefore, firm A's payoff π_A under (p, q^*) satisfies

$$\begin{aligned}
\pi_A(p, q^*, \sigma) &= \sum_{i \in D_A} (p_i - c) \\
&< \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* - c\right) \\
&\leq \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k}\right) + \delta.
\end{aligned} \tag{76}$$

We will show that $\pi_A(p, q^*, \sigma) \leq 0$ for any p by considering the following two cases separately.

Suppose first that $D_A \subsetneq I$ so that $I \setminus D_A \neq \emptyset$. Since the right-hand side of (76) is continuous in ε, if we show that it is less then $-h$ under exact linearity, then $\pi_A(p, q^*) < 0$ holds under approximate linearity. Under exact linearity, (76) becomes

$$\pi_A(p, q^*, \sigma) < \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} \right) + \delta$$
$$= h \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(2\alpha_i^k - d_i \right).$$

Since $\sum_{k=1}^{K} \sum_{i \in D_A^k} \alpha_i^k = L(D_A)$ and $\sum_{k=1}^{K} \sum_{i \in D_A^k} d_i = 2L(D_A) + L(D_A, I \setminus D_A)$, we have

$$\pi_A(p, q^*, \sigma) < -hL(D_A, I \setminus D_A) \leq -h,$$

where the inequality follows since $I \setminus D_A \neq \emptyset$.

Suppose next that $D_A = I$. In this case, $\sum_{i \in D_A} (q_i^* - c) = 0$ by definition so that the definition of δ implies

$$\pi_A(p, q^*, \sigma) = \sum_{i \in D_A} (p_i - c) \leq \sum_{k=1}^{K} \sum_{i \in D_A^k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* - c \right) \leq \delta = \pi_A(p^*, q^*, \sigma).$$

□

References

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