Bertrand Competition under Network Externalities*

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Abstract

Two sellers engage in price competition to attract buyers located on a network. The value of the good of either seller to any buyer depends on the number of neighbors on the network who consume the same good. For a generic specification of consumption externalities, we show that an equilibrium price equals the marginal cost if and only if the buyer network is complete or cyclic. When the externalities are approximately linear in the size of consumption, we identify the class of networks in which one of the sellers monopolizes the market, or the two sellers segment the market.

Key words: graphs, networks, externalities, Bertrand, divide and conquer, discriminatory pricing, monopolization, segmentation, two-sided market.
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1 Introduction

Goods have network externalities when their value to each consumer depends on the consumption decisions of other consumers. The externalities may derive from physical connection to consumers adopting the same good as in the case of telecommunication devices, from provision of complementary goods as in the case of operating systems and softwares for computers, or from pure psychological factors as in the case of a consumption bandwagon. Despite their importance in reality, we only have limited understanding of network externalities particularly when those goods

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are supplied competitively. The objective of this paper is to study price competition in the presence of consumption externalities represented by a buyer network. Specifically, we formulate a model of price competition under local network externalities by supposing that two sellers compete for a network of buyers who experience positive externalities when their neighbors in the network consume the same good.

A more detailed description of our model is as follows: Two sellers each sell goods that are incompatible with each other. Consumers of either good experience larger positive externalities when more of his neighbors in the network consume the same good. In stage 1, the two sellers post prices simultaneously. The prices can be perfectly discriminatory and can be negative. Upon publicly observing the price vectors posted by both sellers, the buyers in stage 2 simultaneously decide which good to buy or not to buy either. The sellers have no cost of serving the market, and their payoffs simply equal the sum of prices offered to the buyers who choose to buy their goods.

In this framework, we find that the equilibrium outcome of price competition subtly depends on the network structure. Our first observation concerns the validity of marginal cost pricing. When no network externalities are present, it is clear that the unique subgame perfect equilibrium of this game has both sellers offer zero to all buyers. We first show that such marginal-cost pricing is consistent with an equilibrium in an arbitrary network when the externalities are linear in the number of neighbors consuming the same good. We note however that while linearity is an important class, it is not a generic property in the space of all externalities. Under a generic specification of externalities, we show the following: (1) Unless the network is complete or cyclic, there exists no equilibrium in which either seller monopolizes the market by offering the same price to all buyers;[1] (2) Unless the network is complete or cyclic, there exists no (monopolization or segmentation) equilibrium in which both sellers offer zero (= marginal cost) to every buyer; (3) If the network is complete or cyclic, there exists an equilibrium in which both sellers offer zero to every buyer and one of them monopolizes the market. We find it surprising that the non-existence results apply even to networks that are symmetric with respect to all buyers. It is also interesting to note that no unintuitive conclusion results as long as we confine ourselves to complete networks, which correspond to global externalities. Given these results, we proceed to the characterization of an equilibrium when the

[1] A graph is complete if any pair of buyers are neighbors. The linear externalities in particular imply that the value of the good is zero to a buyer when none of his neighbors consumes it.

externalities are non-linear.

Positive identification of an equilibrium is possible when the externalities are approximately linear and when the network satisfies certain properties as follows. First, we consider *bipartite* networks. A network is bipartite if the set of buyers is divided into two subsets and if all neighbors of any buyer in one subset belong to the other subset. This class of networks is important since it is a graph-theoretic representation of a *two-sided market* that has received much attention in the literature as discussed in the next section. We show that in a bipartite network, there exists an equilibrium in which one of the sellers monopolizes the market (*i.e.,* buyers on both sides) by charging positive prices to all buyers on one side while subsidizing all buyers on the other side. Furthermore, the equilibrium price to each buyer (either positive or negative) is shown to be approximately proportional to the number of links he has to the other side of the market. We relate these findings to the identification of the buyers that need to be protected from the inducement from the other seller, and those that can be squeezed for profits.

Next, we consider the possibility of a segmentation equilibrium. We say that a network is *bilocular* if the set of buyers is divided into two subsets and if every buyer in each subset has at least as many neighbors in the same subset as in the other subset, and some buyer in each subset has strictly more neighbors in the same subset than in the other subset. When the externalities are approximately linear, we show that market segmentation in a bilocular network takes place in equilibrium with each seller making positive profits.

The paper is organized as follows: After discussing the related literature in the next section, we formulate a model of price competition in Section 3. Section 4 considers the subgame played by the buyers that follows the public observation of prices posted by both sellers. The subgame following each price profile is one of strategic complementarities, and hence has maximal and minimal Nash equilibria. We use the iterated elimination of dominated actions to characterize those equilibria and also to identify the existence of profitable deviations by a seller in the subsequent analysis. We turn to the analysis of a subgame perfect equilibrium in Section 5 and identify lower bounds on the sellers' equilibrium payoffs. Section 6 examines the validity of uniform pricing and marginal cost pricing in equilibrium. With the definition of approximate linearity, we study in Section 7 the possibility of market monopolization in a bipartite network, which corresponds to a two-sided market. Equilibrium market segmentation in bilocular networks is studied in Section 8. We conclude with a discussion in Section 9. The Appendix contains most of the proofs.

2 Related Literature

Since the pioneering work of Dybvig and Spatt (1983), problems related to the monopoly provision of a good with network externalities are studied by Cabral et al. (1999), Park (2004), Sekiguchi (2009), Ochs and Park (2010), Aoyagi (2013), Parakhonyak and Vikander (2013), among others. In light of the multiplicity of equilibria under externalities, these papers study such issues as implementing efficient or revenue maximizing equilibria under complete and incomplete information, intertemporal patterns of adoption decisions, as well as the validity of introductory pricing.[2]

One key ingredient of the present paper is that of *divide-and-conquer,* which has been studied by Segal (2003), Winter (2004) and Bernstein and Winter (2012) among others in contracting problems under externalities. In these problems, a single principal offers a contract to the set of agents whose participation decisions create externalities to other agents. The *divide-and-conquer* strategy of the principal specifies the sequential order in which the principal approaches those agents. The contract offered to the first agent makes it a dominant strategy to accept it even if all other agents reject, the contract offered to the second agent makes it a dominant strategy to accept it even if all but the first agent reject, and so on.[3] Our analysis of an equilibrium involves the same argument: Given some price profile, we examine if it is profitable for either seller to deviate by offering an alternative price vector. We consider price vectors that amount to approaching the buyers one by one in some order and switching them provided that it succeeds in switching all their predecessors. We relate the process to the iterative elimination of strictly dominated strategies, and use it derive a lower bound on equilibrium payoffs.

Modeling local externalities as a graph-theoretic network, Candogan *et al.* (2012) and Bloch and Quérou (2013) both study the problem of optimal monopoly pricing. Candogan *et al.* (2012) formulate a model in which the good is divisible and the

[2]See Rohlfs (1974) for an early treatment of network externalities.

[3]A similar idea can be found in the study of an optimal marketing strategies under externalities in Hartline *et al.* (2008). A marketing strategy determines the order in which the monopolist approaches the set of buyers with private valuations as well as a sequence of contingent prices offered to them. See also Aoyagi (2010) for the analysis of an optimal marketing strategy against informationally interdependent buyers.

externalities between any pair of consumers may be directed. Bloch and Quérou (2013) construct a model in which the good is indivisible and the externalities are undirected, but each consumer has private information about his valuation of the good. It is interesting to note that in both these models, the optimal price is *independent* of the network configuration in the case of undirected externalities as in the present paper, and is *uniform* across the buyers if they have (ex ante) the same valuation.

Competition between suppliers of goods with network externalities was first formulated by Katz and Shapiro (1985). Subsequent work on the subject includes Sundararajan (2003), Gabszewicz and Wauthy (2004), Hagiu (2006), Ambrus and Argenziano (2009), Bernaji and Dutta (2009), Blume *et al.* (2009), Fjeldstad *et al.* (2010), Cabral (2011), Jullien (2011), and Bloch and Quérou (2013). Among them, Blume *et al.* (2009) and Bloch and Quérou (2013) study price competition under local network externalities when market segmentation amongst the sellers is exogenously given.

Ambrus and Argenziano (2009) and Jullien (2011) present models that are most closely related to the present paper. These models are couched in terms of *two-sided markets,* where the sellers are providers of *platforms* who offer a marketplace for agents on two sides such as sellers and buyers of some good. In such models, the utility of an agent on one side is an increasing function of the number of participants from the other side.[4] Ambrus and Argenziano (2009) analyze Bertrand competition between platforms in a two-sided market. Jullien (2011) applies the divide-and-conquer argument to his analysis of multi-sided markets, and derives a bound on the platforms' payoffs when they engage in Stackelberg price competition. Both Ambrus and Argenziano (2009) and Jullien (2011) formulate externalities differently from the present paper, and also make some assumptions on the ability of the agents to coordinate their actions. For example, the assumption of correlated rationalizability by Ambrus and Argenziano (2009) implies that the agents coordinate on the pareto-efficient alternative whenever there is one. In contrast, our interest is in the maximal scope of an equilibrium when there is no restriction on the buyers' strategies. Specifically, our argument is based on the bang-bang property of a subgame perfect equilibrium by allowing full coordination by the buyers on an extreme equilibrium following any deviation by either seller.

Banerji and Dutta (2009) use the graph-theoretic definition of network exter-

[4] See Armstrong (1998), and Laffont et al. (1998a,b).

nalities as in the present paper, and identify conditions under which price competition leads to monopolization and market segmentation. They assume that the sellers cannot price discriminate the buyers, and also place restrictions on the buyers' strategies. These differences in assumptions make it difficult to compare their findings with ours.

3 Model

Two sellers A and B compete for the set $I = \{1, \ldots, N\}$ of $N \geq 3$ buyers. Consumption of either seller's good generates externalities to the buyers according to a buyer network. Formally, a buyer network is represented by a simple undirected graph G whose nodes correspond to the buyers, and consumption externalities exist between buyers i and j if they are *adjacent* in the sense that there is a link between i and j. When buyer j is adjacent to buyer i, we also say that j is i's *neighbor.*

The buyer network G is *connected* in the sense that for any pair of buyers i and j, there exists a path from i to j. That is, there exist buyers $i_1, i_2, \ldots, i_m$, such that i_1 is adjacent to i, i_2 is adjacent to i_1, $\ldots$, and i_m is adjacent to j. For any buyer i in network G, denote by $N_i(G)$ (or simply N_i) the set of i's neighbors in G. The *degree* $d_i(G) = |N_i(G)|$ of buyer i in network G is the number of i's neighbors. Define also M to be the number of links in G. Since each link counts twice when aggregating the number of degrees in G, we have $M = \frac{1}{2} \sum_{i \in I} d_i$.

For $r = 2, \ldots, N-1$, the network G is *r-regular* if all buyers have the same degree r, and regular if it is r-regular for some r. G is *cyclic* if it is connected and 2-regular, and *complete* if it is $(N-1)$-regular, or equivalently, every pair of buyers are adjacent to each other.

The value of either seller's good to any buyer i is determined by the number of neighbors of i who consume the same good. We denote by v^n the value of either good to any consumer when n of his neighbors consume the same good. In particular, v^0 denotes the *stand-alone value,* or the value to any buyer of either good when none of his neighbors consumes the same good. Implicit in this assumption is that the two goods A and B are incompatible with each other since the value of either good to any buyer is assumed the same whether his neighbor consumes the other good or nothing. The value does not depend on the identity of a buyer or the identity of the seller who supplies the good. The consumption externalities are non-negative in the sense that $0 \leq v^0 \leq v^1 \leq \cdots \leq v^{N-1}$.

The good can be produced at no cost for both sellers. The sellers can perfectly price discriminate the buyers, and we let p_i and q_i denote the prices offered to buyer i by seller A and seller B, respectively. They simultaneously quote price vectors $p = (p_i)_{i \in I} \in \boldsymbol{R}^N$ and $q = (q_i)_{i \in I} \in \boldsymbol{R}^N$. The buyers publicly observe (p, q), and then simultaneously decide whether to buy either good, or buy neither.

Public observability of the entire price vectors and the possibility of perfect price discrimination are the two key assumptions of our model. We note in passing that these assumptions may be more in line with the reality for intermediate goods markets with a limited number of buyers than for large consumption goods markets.[5]

Buyer i's action x_i is an element of the set $S_i = \{A, B, \emptyset\}$, where $\emptyset$ represents no purchase. Each seller's strategy is an element of $\boldsymbol{R}^N$, whereas buyer i's strategy σ_i is a mapping from the set $\boldsymbol{R}^{2N}$ of price vectors (p, q) to S_i. For each action profile $x = (x_i)_{i \in I} \in S = \prod_{i \in I} S_i$ of buyers, let

$$I_A(x) = \{i \in I : x_i = A\}, \quad \text{and} \quad I_B(x) = \{i \in I : x_i = B\}$$

denote the set of buyers choosing A and the set of buyers choosing B, respectively. Given the price profile (p, q), buyer i's payoff under the action profile x is given by

$$u_i(x) = \begin{cases} v^{|N_i \cap I_A(x)|} - p_i & \text{if } x_i = A, \\ v^{|N_i \cap I_B(x)|} - q_i & \text{if } x_i = B, \\ 0 & \text{if } x_i = \emptyset, \end{cases} \tag{1}$$

If we denote by $\sigma = (\sigma_i)_{i \in I}$ the buyers' strategy profile, the payoffs $\pi_A(p, q, \sigma)$ and $\pi_B(p, q, \sigma)$ of sellers A and B, respectively, under the strategy profile (p, q, σ) are given by

$$\pi_A(p, q, \sigma) = \sum_{i \in I_A(\sigma(p,q))} p_i,$$
$$\pi_B(p, q, \sigma) = \sum_{i \in I_B(\sigma(p,q))} q_i,$$

and buyer i's payoff $\pi_i(p, q, \sigma)$ under the strategy profile (p, q, σ) is given by

$$\pi_i(p, q, \sigma) = u_i(\sigma(p, q)).$$

A price vector (p^*, q^*) and a strategy profile $\sigma = (\sigma_i)_{i \in I}$ together constitute a *subgame perfect equilibrium* (SPE) if given any price vector $(p, q) \in \boldsymbol{R}^{2N}$, the action

[5]For example, price discrimination in large markets may be better modeled as one full price and one discount price as in Candogan *et al.* (2012).

vector $(\sigma_i(p,q))_{i\in I}$ is a Nash equilibrium of the subgame following (p,q), and given σ, each component of the price vector (p^*,q^*) is optimal against the other:

$$\pi_i\left(p,q,\sigma(p,q)\right) \geq \pi_i\left(p,q,x_i,\sigma_{-i}(p,q)\right) \text{ for every } x_i,\ i \text{ and } (p,q),$$
$$\pi_A\left(p^*,q^*,\sigma(p^*,q^*)\right) \geq \pi_A\left(p,q^*,\sigma(p,q^*)\right) \text{ for every } p,$$
$$\pi_B\left(p^*,q^*,\sigma(p^*,q^*)\right) \geq \pi_B\left(p^*,q,\sigma(p^*,q)\right) \text{ for every } q.$$

4 Nash Equilibrium in the Buyers' Game

In this section, we fix the price vector (p,q), and consider an equilibrium of the buyers' subgame following (p,q) in which the set of actions of each buyer i equals $S_i = \{A, B, \emptyset\}$, and his payoff function u_i is defined by (1). The simultaneous-move game $(I, S = \prod_{i\in I} S_i, (u_i)_{i\in I})$ among the buyers is a supermodular game when the set S_i of actions of each buyer is endowed with the ordering $A \succ \emptyset \succ B$. It follows that the game has pure Nash equilibria that are maximal and minimal with respect to the partial ordering $\succ_S$ on S induced by $\succ$.[6] We refer to the maximal equilibrium as the *A-maximal* equilibrium and denote it by x^A, and the minimal equilibrium as the *B-maximal* equilibrium and denote it by x^B. By definition, for any NE y and buyer i, $y_i = A$ implies $x_i^A = A$, and $y_i = B$ implies $x_i^B = B$.

It is known that any NE must survive the iterative elimination of strictly dominated actions, and that in a finite supermodular game, any strategy profile x that survives this process lies between x^A and x^B: $x^A \succcurlyeq_S x \succcurlyeq_S x^B$.[7] In what follows, we apply the iterative elimination process to the buyers' game and use it to characterize the maximal and minimal NE. The notation appearing in this process will be used in the subsequent analysis.

Define $T_0 = \emptyset$ and $S_0 = S$, and suppose that for $k = 1, 2, \ldots$, the set $T_{k-1} \subset I$ and the action profile $x^*_{T_{k-1}}$ of buyers in T_{k-1} have been specified. Intuitively, T_{k-1} is the set of buyers i for whom x_i^* has been identified as a dominant action after $k-1$ rounds of elimination of strictly dominated actions. Formally, for any product subset $S' = \prod_i S'_i \subset S$ of action profiles such that $S'_i \neq \emptyset$, buyer i's action $x_i \in S'_i$ is (strictly) *dominated* in S' (by another pure action) if there exists $x'_i \in S'_i$ such that

$$u_i(x_i, x_{-i}) < u_i(x'_i, x_{-i}) \text{ for every } x_{-i} \in S'_{-i}.$$

$x \in S'_i$ is *dominant* in S' if any other action $x'_i \in S'_i$ is dominated in S' (by x).

[6] See Topkis (1998).
[7] See Milgrom and Roberts (1990).

For $k = 1, 2, \ldots$, let

$$\begin{aligned} Y_k &= \left\{ i \in I \setminus T_{k-1} : x_i = A \text{ is dominated in } S^{k-1} \right\}, \\ Z_k &= \left\{ i \in I \setminus T_{k-1} : x_i = B \text{ is dominated in } S^{k-1} \right\}, \\ W_k &= \left\{ i \in I \setminus T_{k-1} : x_i = \emptyset \text{ is dominated in } S^{k-1} \right\}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} S^k = \{ x \in S^{k-1} : x_j \neq A \text{ if } j \in Y_k,\ x_j \neq B \text{ if } j \in Z_k \\ \text{and } x_j \neq \emptyset \text{ if } j \in W_k \}. \end{aligned} \tag{3}$$

Further, let

$$\begin{aligned} P_k = Y_k \cap W_k &= \left\{ i \in I \setminus T_{k-1} : x_i = B \text{ is dominant in } S^{k-1} \right\}, \\ Q_k = Z_k \cap W_k &= \left\{ i \in I \setminus T_{k-1} : x_i = A \text{ is dominant in } S^{k-1} \right\}, \\ R_k = Y_k \cap Z_k &= \left\{ i \in I \setminus T_{k-1} : x_i = \emptyset \text{ is dominant in } S^{k-1} \right\}. \end{aligned} \tag{4}$$

Define now

$$T_k = T_{k-1} \cup (P_k \cup Q_k \cup R_k), \tag{5}$$

and

$$x_i^* = \begin{cases} B & \text{if } i \in P_k, \\ A & \text{if } i \in Q_k, \\ \emptyset & \text{if } i \in R_k. \end{cases} \tag{6}$$

Since each buyer has at most two dominated actions, the above process stops in or before $2N$ rounds. Let then K be the minimal number such that

$$P_{k+1} = Q_{k+1} = R_{k+1} = \emptyset \text{ for } k \geq K.$$

In other words, no buyer has a dominant action in S^k for $k \geq K$. The sets S^k, T_k, Y_k, Z_k, W_k, P_k, Q_k, and R_k as well as the number K all depend on the price profile (p, q). In this sense, we write $Q_{k+1}(p, q)$ and so on when we want to make this dependence explicit.

If $x \in S$ is any NE, every buyer in T_K must be choosing his iteratively dominant action in x so that

$$x_{T_K} = x_{T_K}^*.$$

It follows that any two NE may be different from each other only in the actions chosen by buyers in $I \setminus T_K$. The following proposition states that the A-maximal and B-maximal NE can be constructed by having the maximal number of buyers among them choose A and B, respectively. Specifically, let $J_A \subset I \setminus T_K$ be the maximal set that satisfies

$$u_i\left(x^*_{T_K}, x_{J_A} = (A, \ldots, A), x_{I \setminus T_K \setminus J_A} = (\emptyset, \ldots, \emptyset)\right) \geq 0.$$

Note that the maximality is well-defined since if the inequality holds for J and $J' \subset I \setminus T_K$, then it also holds for $J \cup J'$. The buyers in J_A can each realize a non-negative payoff by collectively choosing A.[8] Likewise, let $J_B \subset I \setminus T_K$ be the maximal set that satisfies

$$u_i\left(x^*_{T_K}, x_{J_B} = (B, \ldots, B), x_{I \setminus T_K \setminus J_B} = (\emptyset, \ldots, \emptyset)\right) \geq 0.$$

Proposition 1 *Define x^A and x^B by*

$$x^A = (x^*_{T_K}, x_{J_A} = (A, \ldots, A), x_{I \setminus T_K \setminus J_A} = (\emptyset, \ldots, \emptyset)), \quad \textit{and}$$
$$x^B = (x^*_{T_K}, x_{J_B} = (B, \ldots, B), x_{I \setminus T_K \setminus J_B} = (\emptyset, \ldots, \emptyset)).$$

Then x^A and x^B are the A-maximal and B-maximal NE, respectively.

Proof. We show that x^A is an A-maximal NE. The symmetric argument shows that x^B is a B-maximal NE. In particular, when $T_K = I$, every buyer has an iteratively dominant action, and $x^A = x^B$ is the unique NE.

– x^A is a NE.

In x^A, any buyer $i \in T_K$ is choosing his iteratively dominant action and hence has no incentive to deviate. Take $i \in I \setminus T_K$. If $i \in J_A$, then since $x_i = A$ yields by definition a non-negative payoff to buyer i, he cannot profitably deviate to $x_i = \emptyset$. If i can profitably deviate to $x_i = B$, then then $x_i = B$ would be his dominant action in S_K since no other buyer in $I \setminus T_K$ chooses B in x^A. This would be a contradiction to $P_{K+1} = \emptyset$. If $i \notin J_A$, then $x_i = A$ is not a profitable deviation for buyer i since if it were, then we would have a contradiction to the maximality of J_A. $x_i = B$ is not a profitable deviation either since if it were, then we would have a contradiction to $P_{K+1} = \emptyset$ by the same logic as above.

– x^A is A-maximal.

[8] Set J_A can alternatively obtained by eliminating $x_i = A$ if it is iteratively dominated by $x_i = \emptyset$ in S_K.

Take any NE x. As noted in the text, $x_{T_K} = x^*_{T_K} = x^A_{T_K}$. If $x_i = A$ for $i \in I \setminus T_K \setminus J_A$, then x cannot be a NE since $x_i = \emptyset$ would be a profitable deviation for him by the definition of J_A. It follows that no NE x can have more buyers choose A than x^A. ■

The dominance argument can be described more explicitly in terms of v^d and (p, q) as follows. Note that the minimal number of i's neighbors who may choose A in S^{k-1} is given by

$$\alpha_i^k = \left| N_i \cap \left\{ j : S_j^{k-1} = \{A\} \right\} \right|,$$

and that the maximal number of i's neighbors who may choose B in S^{k-1} is given by

$$\beta_i^k = \left| N_i \cap \left\{ j : B \in S_j^{k-1} \right\} \right|.$$

It follows that $x_i = A$ is dominant in S^{k-1} (*i.e.*, $i \in Q_k(p, q)$) if and only if

$$v^{\alpha_i^k} - p_i > \max \left\{ v^{\beta_i^k} - q_i, 0 \right\},$$

or equivalently,

$$p_i < \max \left\{ v^{\alpha_i^k} - v^{\beta_i^k} + q_i^*, v^{\alpha_i^k} \right\}. \tag{7}$$

This is the key inequality that will be used extensively in what follows.

5 Subgame Perfect Equilibrium

We now turn to the original two-stage game including the sellers. The proposition below makes a simple observation that if a price vector (p^*, q^*) is sustained in some SPE, then it must be sustained in an SPE in which the buyers choose an extreme response to either seller's deviation: If seller A deviates from p^*, then all buyers coordinate on the B-maximal NE that least favors seller A, and vice versa. The proposition hence presents a bang-bang property of an SPE.

Proposition 2 *For any network G, (p^*, q^*) is an SPE price vector if and only if there exists buyers' strategy profile σ such that (p^*, q^*, σ) is an SPE and*

$$\sigma(p, q) = \begin{cases} \sigma^B(p, q) & \text{if } p \neq p^* \text{ and } q = q^*, \\ \sigma^A(p, q) & \text{if } p = p^* \text{ and } q \neq q^*. \end{cases}$$

Consider next seller A's best response p to B's price q when the buyers play the B-maximal strategy σ^B. Since $\sigma^B(p,q)$ is a B-maximal NE for any (p,q), seller A can attract buyer i if and only if $x_i = A$ is an iteratively dominant action for buyer i: $i \in \cup_{k=1}^K Q_k$, where Q_k is as defined in (4). Hence,

$$\pi_A(p,q,\sigma^B) = \sum_{k=1}^{K} \sum_{i \in Q_k} p_i.$$

The following lemma shows that if seller A's price vector p is a best response to (q,σ^B), then no two buyers in $Q_k = Q_k(p,q)$ are adjacent, where Q_k is as defined in (4) and equals the set of buyers for whom A is dominant in round $k-1$ of the iteration process. In other words, the optimal way to attract adjacent buyers i and j is to approach them sequentially. Intuitively, this is because making choice A dominant for both buyers simultaneously requires offering lower prices to both of them than making $x_i = A$ dominant for buyer i first, then making $x_j = A$ dominant for buyer j next conditional on i choosing $x_i = A$.

Lemma 3 *Let $(Q_k)_{k=1,\ldots,K}$ be as defined in (4) under the price vector (p,q). If p is a best response to (q,σ^B), then for every $k = 1,\ldots,K$,*

$$i,j \in Q_k \quad \Rightarrow \quad i \text{ and } j \text{ are not adjacent.}$$

We now derive a key result that establishes a lower bound for each seller's equilibrium payoff given the price vector of the other seller. Although the discussion is based on the iterated dominance argument of Section 4, we find it useful to present it in terms of the sequence of buyers rather than the sequence of sets of buyers. We return to the comparison of the two processes later in the section. As mentioned in the Introduction, the argument is one of *divide and conquer,* where seller A, say, approaches each buyer sequentially according to some ordered list, and presents them with a price which makes the choice A a dominant action when all his predecessors in the list choose A.

Formally, fix the price q^* of seller B, and suppose that the buyers play the B-maximal NE facing (p,q^*) for any p: Buyer i chooses $x_i = A$ only when it is an iteratively dominant action. Suppose further that seller A approaches the buyers in the order $i_1,\ldots,i_N$: Seller A first makes a price offer to buyer i_1 that makes A

dominant for him. In fact, $x_{i_1} = A$ is dominant for buyer i_1 if p_{i_1} is such that

$$v^0 - p_{i_1} > \max\left\{v^{d_{i_1}} - q^*_{i_1},\, 0\right\},$$

or equivalently,

$$p_{i_1} < \min\left\{v^0 - v^{d_{i_1}} + q^*_{i_1},\, v^0\right\}.$$

Let $H_1 = \{i_1\}$. Seller A next makes a price offer to buyer i_2 that makes $x_{i_2} = A$ dominant given the choice of buyer i_1. This can be accomplished by p_{i_2} such that

$$p_{i_2} < \min\left\{v^{s_{i_2}} - v^{d_{i_2} - s_{i_2}} + q^*_{i_2},\, v^{s_{i_2}}\right\},$$

where $s_{i_2} = |N_{i_2} \cap H_1|$ so that $s_{i_2} = 1$ if buyer i_2 is adjacent to i_1, and $= 0$ otherwise. Now let $H_2 = \{i_1, i_2\}$. Proceeding iteratively, we see that seller A can have buyer i_k choose $x_{i_k} = A$ as his iteratively dominant action by offering p_{i_k} such that

$$p_{i_k} < \min\left\{v^{s_{i_k}} - v^{d_{i_k} - s_{i_k}} + q^*_{i_k},\, v^{s_{i_k}}\right\}, \tag{8}$$

where $s_{i_k} = |N_{i_k} \cap H_{k-1}|$ is the number of neighbors of i_k in the set $H_{k-1} = \{i_1, \ldots, i_{k-1}\}$. Intuitively, s_{i_k} is the externalities of good A to buyer i_k when those buyers in H_{k-1} choose A. On the other hand, $d_{i_k} - s_{i_k}$ gives an upper bound on the externalities of good B to i_k when only those buyers in $I \setminus H_{k-1}$ may choose B. Note that for any list $i_1, \ldots, i_N$ of buyers,

$$\sum_{k=1}^{N} s_{i_k} = M,$$

where M is the total number of links in G. Define S by

$$S = \Big\{ s = (s_i)_{i \in I} : s_{i_1} = 0 \text{ and } s_{i_k} = |N_{i_k} \cap \{i_1, \ldots, i_{k-1}\}| \text{ for } k \geq 2 \text{ for some ordering } (i_1, \ldots, i_N) \text{ of buyers} \Big\}. \tag{9}$$

Note that if s corresponds to the list $i_1, \ldots, i_N$, then $d - s = (d_i - s_i)_{i \in I}$ corresponds to the reversed list $i_N, \ldots, i_1$. Hence, if $s \in S$, then $d - s \in S$ as well.

Some comments are in order on the above process of divide and conquer. First, in relation to the iterated dominance argument of Section 4, buyer i_1 belongs to $Q_1(p, q^*)$ defined in (4) since he has a dominant action in $S^0 = S$. Buyer i_2 belongs to Q_2 if he is adjacent to i_1 since then $x_{i_2} = A$ is dominant only after $x_{i_1} = B$ and $x_{i_1} = \emptyset$ are eliminated from S^0. Otherwise, $x_{i_2} = A$ is dominant in S^0 itself so that

$i_2 \in Q_1$ as well. In general, buyer i_k belongs to one of $Q_1, Q_2, \ldots, Q_k$ depending on the status of his neighbors. In other words,

$$H_k \subset \cup_{\ell=1}^{k} Q_\ell(p, q^*).$$

Next, against some price vector q^* of seller B, seller A may achieve a higher payoff by offering prices that attract only a subset of buyers than offering prices that attract all of them. The above process to the contrary assumes that seller A attracts all buyers by offering p. In other words, we use the existence of such a price vector p to establish a necessary condition for an equilibrium: (p^*, q^*, σ) is an equilibrium only if $\pi_A(p, q^*, \sigma) > \pi_A(p^*, q^*, \sigma)$ for any p that attracts all buyers.

To summarize the discussion so far, even if the buyers play the B-maximal equilibrium $\sigma^B(p, q^*)$ that least favors seller A, he can attract all buyers by offering the prices satisfying (8). We hence have the following lemma that gives a lower bound for each seller's equilibrium payoff.

Lemma 4 *If (p^*, q^*, σ) is an SPE, then*

$$\begin{aligned} \pi_A(p^*, q^*, \sigma) &\geq \max_{s \in S} \sum_{i=1}^{N} \min\left\{v^{s_i} - v^{d_i - s_i} + q_i^*,\, v^{s_i}\right\}, \\ \pi_B(p^*, q^*, \sigma) &\geq \max_{s \in S} \sum_{i=1}^{N} \min\left\{v^{s_i} - v^{d_i - s_i} + p_i^*,\, v^{s_i}\right\}. \end{aligned} \tag{10}$$

While the above lemma gives a lower-bound, note also that (8) implies the following inequality on seller A's payoff from any given divide-and-conquer pricing strategy p:

$$\sum_{i=1}^{N} p_i < \sum_{i=1}^{N} \min\left\{v^{s_i} - v^{d_i - s_i} + q_i^*,\, v^{s_i}\right\}. \tag{11}$$

We use (11) as a way to explain intuition for some of the results in what follows.

Figures 1 and 2 illustrate the discussion for the line network of three buyers. In Figure 1, seller A approaches the buyers in the order $(i_1, i_2, i_3) = (1, 3, 2)$ while seller B offers $q^* = (q_1^*, q_2^*, q_3^*)$: Seller A can make $x_1 = A$ dominant actions for buyer 1 if his payoff from choosing A is strictly higher than that from choosing either $\emptyset$ or B under the assumption that his neighbor (*i.e.*, buyer 2) chooses B. This leads to the comparison between $v^0 - p_1$ and $\max\left\{v^1 - q_1^*, 0\right\}$. The same argument applies to buyer 2. When p satisfies the stated inequalities, hence, $Q_1 = \{1, 3\}$ since

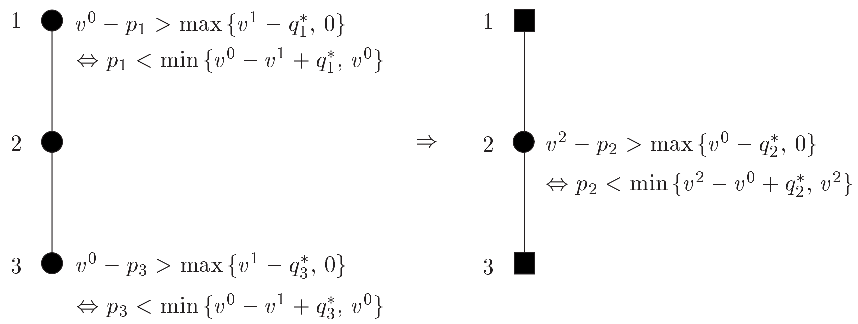

Figure 1: Divide-and-conquer by seller A with $(i_1, i_2, i_3) = (1, 3, 2)$.

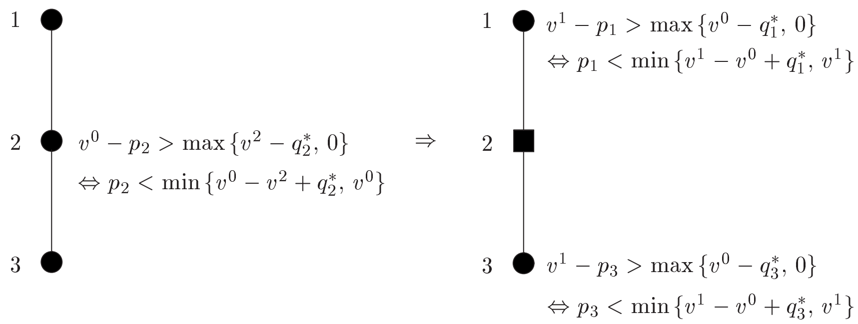

Figure 2: Divide-and-conquer by seller A with $(i_1, i_2, i_3) = (2, 1, 3)$.

$x_i = B$ and $x_i = \emptyset$ are eliminated in the first round in the iterated elimination process for both $i = 1$ and $i = 3$. For buyer 2, on the other hand, $x_2 = A$ is a dominant action for him if his payoff from choosing A is strictly higher than that from choosing either $\emptyset$ or B under the assumption that his neighbors (*i.e.*, buyers 1 and 3) choose A. This leads to the comparison between $v^2 - p_2$ and $\max\left\{v^0 - q_2^*, 0\right\}$. Under the stated inequalities, hence, $Q_2 = \{2\}$. Hence, even if the buyers play the B-maximal equilibrium $\sigma^B(p, q^*)$, seller A's divide-and-conquer strategy with $(i_1, i_2, i_3) = (1, 3, 2)$ is a profitable deviation if

$$\begin{aligned} \min\{v^0 - v^1 + q_1^*, v^0\} + \min\{v^0 - v^1 + q_3^*, v^0\} \\ + \min\{v^2 - v^0 + q_2^*, v^2\} > \pi_A(p^*, q^*, \sigma). \end{aligned} \tag{12}$$

Likewise, his divide-and-conquer strategy with $(i_1, i_2, i_3) = (2, 1, 3)$ illustrated in Figure 2 is a profitable deviation if

$$\begin{aligned}\min\{v^0 - v^2 + q_2^*,\, v^0\} + \min\{v^1 - v^0 + q_1^*,\, v^1\}& \\ + \min\{v^1 - v^0 + q_3^*,\, v^1\} &> \pi_A(p^*, q^*, \sigma),\end{aligned} \tag{13}$$

and that with $(i_1, i_2, i_3) = (1, 2, 3)$ is a profitable deviation if

$$\begin{aligned}\min\{v^0 - v^1 + q_1^*,\, v^0\} + \min\{v^1 - v^1 + q_2^*,\, v^1\}& \\ + \min\{v^1 - v^0 + q_3^*,\, v^1\} &> \pi_A(p^*, q^*, \sigma).\end{aligned} \tag{14}$$

It follows that (p^*, q^*) cannot be an equilibrium price vector if any one of the inequalities (12), (13) and (14) holds. This will be examined for the price vector $(p^*, q^*) = (0, 0)$ in the next section.

6 Uniform and Marginal-Cost Pricing

When there are no consumption externalities $0 < v^0 = \cdots = v^{N-1}$, it is clear that a subgame perfect equilibrium price (p^*, q^*) is unique and equal to the marginal cost: $(p^*, q^*) = (0, 0)$. In this section, we will examine if and how this result can be extended when there are externalities.

Let $D = D(G)$ be the highest degree in G:

$$D(G) = \max_{i \in I} d_i(G).$$

For the network G, hence, the relevant levels of externalities are $(v^0, \ldots, v^D)$. We say that the externalities $(v^0, \ldots, v^D)$ are *linear* if there exists $h > 0$ such that

$$v^k = kh \quad \text{for every } k = 0, 1, \ldots, D.$$

Note that linearity implies the zero stand-alone value v^0 and hence pure network externalities or pure intermediation. Linearity is a working assumption in many models of network externalities in the literature.[9]

Proposition 5 *Let G be an arbitrary buyer network. Under the linear externalities $(v^0, \ldots, v^D)$, $(p^*, q^*) = (0, 0)$ is an SPE price vector.*

[9]See, for example, Caillaud and Jullien (2003) and Ambrush and Argenziano (2009). On the other hand, linearity violates the weak externalities defined in Jullien (2011, Assumption 1).

To see the intuition behind Proposition 5, note that no divide-and-conquer strategy is profitable under linearity: When seller B monopolizes the market with $q_i^* = 0$ for every i and the buyers play the B-maximal equilibrium following seller A's deviation from 0, (11) shows that his payoff from a divide-and-conquer pricing strategy satisfies

$$\sum_{i=1}^{N} p_i < \sum_{i=1}^{N} \min \left\{ v^{s_i} - v^{d_i - s_i} + q_i^*, 0 \right\} \leq \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right) = 0,$$

where the equality is an immediate consequence of linearity since $\sum_{i=1}^{N} s_i = \sum_{i=1}^{N} (d_i - s_i) = M$ as noted earlier.

We next consider the consequence of introducing some generic property of externalities. As will be seen, whether or not the marginal cost pricing can be an equilibrium depends crucially on the configuration of the buyer network in this case. Specifically, for S defined in (9), suppose that the externalities $(v^0, \ldots, v^D)$ satisfy the following condition:

$$s \in S \text{ and } d - s \text{ is not a permutation of } s \Rightarrow \sum_{i=1}^{N} v^{s_i} \neq \sum_{i=1}^{N} v^{d_i - s_i}. \qquad (15)$$

Recall that s is the sequence of externalities of one good, say A, when the buyers switch to A one by one in some order. $d - s$, on the other hand, is the sequence of externalities of good A when they switch to A one by one in the reverse order. (15) implies that the sum of externalities over buyers is different between the two procedures. The set of $(v^0, \ldots, v^D)$ satisfying (15) is generic in the set

$$\left\{ (v^0, \ldots, v^D) : 0 \leq v^0 \leq \cdots \leq v^D \right\}$$

of all externalities.

Lemma 4 in the preceding section shows that a seller's equilibrium payoff is closely linked to the value of

$$\max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right).$$

It turns out that whether this quantity is positive or not under (15) depends crucially on the network configuration as seen in the following lemma.

Lemma 6 *Suppose that the externalities* $v = (v^0, \ldots, v^D)$ *satisfy (15). If the buyer network* G *is neither cyclic nor complete, then*

$$\max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right) > 0. \tag{16}$$

The proof of Lemma 6 involves showing that by choosing the order $(i_1, \ldots, i_N)$ appropriately, we can always make s and $d - s$ not permutations of each other unless the network is cyclic or complete. In fact, this is accomplished by choosing only the first three buyers (i_1, i_2, i_3) appropriately. The following lemma, which readily follows from Lemmas 4 and 6, provides some key observations on equilibrium pricing.

Lemma 7 *Suppose that* (p^*, q^*, σ) *is an SPE for the buyer network* G *which is neither complete nor cyclic, and that the externalities* $v = (v^0, \ldots, v^D)$ *satisfy (15). Then*

a) $\pi_A(p^*, q^*, \sigma) = 0 \Rightarrow \min_i q_i^* < 0.$

b) $\pi_A(p^*, q^*, \sigma) \leq \sum_i q_i^* \Rightarrow \max_i q_i^* > v^0.$

c) $I_B(\sigma(p^*, q^*)) = I \Rightarrow \max_i q_i^* > v^0,\ \min_i (v^{d_i} - q_i^*) \geq v^0,$ *and* $v^D > 2v^0.$

Note that (a) and (b) of Lemma 7 hold true whether monopolization or segmentation takes place in equilibrium, while (c) applies only to a monopolization equilibrium. An immediate consequence of this lemma is the impossibility of uniform pricing under monopolization: Suppose that monopolization by seller B takes place in equilibrium: $I_B(\sigma(p^*, q^*)) = I$. Then seller B must subsidize at least one buyer by (a), and the price for some buyer is strictly above the stand-alone value by (c):

$$\min_i q_i^* < 0 \leq v^0 < \max_i q_i^* < v^D - v^0. \tag{17}$$

Proposition 8 *Suppose that the buyer network* G *is neither complete nor cyclic and that the externalities* $v = (v^0, \ldots, v^D)$ *satisfy (15). Then there exists no SPE in which one of the sellers monopolizes the market by charging the same price to every buyer.*

It is interesting to note that Candogan *et al.* (2012) and Bloch and Quérou (2013) both find that uniform pricing is optimal for a monopolist when the externalities are undirected as in the present paper. When the sellers face competition, however, it is no longer an equilibrium by Proposition 8. We also note that there are networks which are not cyclic or complete, but are symmetric with respect to every buyer. For example, consider the buyer network in Figure 3. Under a generic specification of externalities, these ex ante symmetric buyers face price discrimination in equilibrium if one of the sellers monopolizes the market.

For monopolization to take place in equilibrium, we also see from (17) that the largest externalities in a network cannot be too small compared with the stand-alone value: $v^D > 2v^0$. This is a non-trivial restriction for networks in which every buyer has a small degree as in line networks.

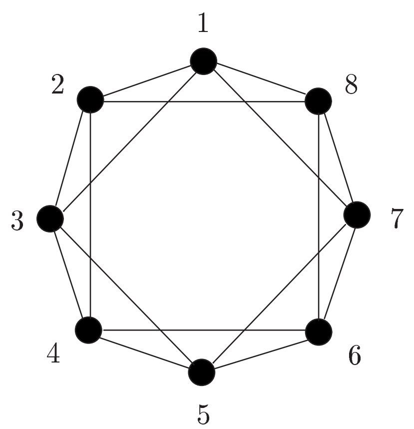

Figure 3: 4-regular network with eight buyers

We are now ready to state our main result on marginal cost pricing. Suppose that both sellers offer zero to all the buyers. In this case, both sellers' payoffs equal zero regardless of whether or not they capture a positive segment of the market. Hence, this price profile cannot be an equilibrium by Lemma 7(a) unless the network is complete or cyclic. The following proposition proves the reverse implication that when the network is either complete or cyclic, there indeed exists an SPE in which both sellers offer zero.

Proposition 9 *Let a buyer network G be given and the externalities $v = (v^0, \dots, v^D)$ satisfy (15). $(p^*, q^*) = (0, 0)$ is an SPE price vector if and only if G is either cyclic or complete.*

The intuition between the possibility of an equilibrium in a cyclic or complete network is as follows. Suppose that seller B monopolizes the market with $q^* = 0$, and that the buyers play the B-maximal equilibrium following (p, q^*). First, in a cyclic network, any buyer i seller A attempts to attract in the first round of the domination process must be offered the price such that $p_i < v^0 - v^2 < 0$ since joining A implies i has no neighbor while remaining at B implies he has two neighbors. On the other hand, seller A can make positive profits only when he attracts a buyer whose both neighbors have already been attracted to A. In this case, A can offer the price such that $p_i < v^2 - v^0$. Whether or not seller A can make positive profits, hence, comes down to the simple comparison between the number of buyers whose both neighbors have already switched to A, and the number of buyers who switch in the first round. Simple inspection shows that the former cannot be greater than the latter. The argument for a complete network is based on a different logic. In a complete network, if seller A employs divide-and-conquer, the order in which he approaches the buyers is immaterial. If seller A attracts buyers $1, \ldots, N$ in this order, then he needs to offer the prices such that $p_1 < v^0 - v^{N-1}$, $p_2 < v^1 - v^{N-2}, \ldots, p_N < v^{N-1} - v^0$. It is then clear that these prices sum up to less than zero.

For illustration of the impossibility of marginal cost pricing, return to the example of the three-buyer line network depicted in Figures 1 and 2. Suppose that $q^* = 0$. In this case, we have

$$\begin{aligned} (12) \quad &\Leftrightarrow \quad 2v^1 - v^2 - v^0 < 0, \\ (13) \quad &\Leftrightarrow \quad 2v^1 - v^2 - v^0 > 0. \end{aligned}$$

Hence, if

$$2v^1 \neq v^2 + v^0, \tag{18}$$

seller A can profitably divide and conquer the buyers against $q^* = 0$. Note that (18) corresponds to (16) in Lemma 6: It fails under the linear externalities $v^0 = 0$, $v^1 = h$ and $v^2 = 2h$, but is true under generic specifications of v^0, v^1 and v^2.

7 Monopolization on a Bipartite Network

The results in the preceding section suggest that some form of discriminatory pricing is inevitable in equilibrium. A natural question then is on the form of equilibrium price discrimination. Interesting related questions are (1) which buyers are the "weak link" in the network that need to be protected, and (2) which buyers can be squeezed for more profits. Since it appears difficult to provide general answers to these questions, we will restrict attention to certain classes of networks for the identification of an equilibrium. In this section, we identify a class of networks in which monopolization takes place in equilibrium.

Our analysis in what follows assumes that the externalities are approximately linear in the following sense: For $h > 0$, the externalities $(v^0, \ldots, v^D)$ are *ε-close to linear* if

$$|v^k - kh| < \varepsilon \text{ for } k = 0, 1, \ldots, D.$$

Since the condition holds for any $\varepsilon > 0$ when the externalities are exactly linear, the conclusions under approximate linearity are valid under exact linearity. In conjunction with Proposition 5, then, this implies the multiplicity of equilibria in these markets.

The buyer network is *bipartite* if the buyer set I is partitioned into two disjoint subsets I_1 and I_2 such that every neighbor of $i \in I_1$ belongs to I_2 and every neighbor of $i \in I_2$ belongs to I_1. Line and star networks are simple examples of a bipartite network. For example, the line network in Figures 1 and 2 is bipartite with the partition $I_1 = \{1, 3\}$ and $I_2 = \{2\}$. A cycle network with an even number of buyers is also bipartite. A bipartite network is *complete* if every buyer in I_1 is linked to every buyer in I_2. Recall that d_i denotes the degree of buyer i. By renaming the partition elements if necessary, we may suppose without loss of generality that I_1 and I_2 satisfy

$$\sum_{i \in I_1} \left(v^{d_i} - v^0 \right) \geq \sum_{i \in I_2} \left(v^{d_i} - v^0 \right). \tag{19}$$

Bipartite networks are particularly important since they represent two-sided markets that attract much attention in the literature. For example, we can think of I_1 as the set of sellers and I_2 as the set of buyers of a certain good. In this case, the sellers A and B are interpreted as the platforms that offer marketplace to these sellers and buyers, and their prices are interpreted as *participation fees* required for registration into their platforms. A complete bipartite network corresponds to a two-sided market in which each agent finds more value in a given platform whenever more agents on the other side participate in the same platform. Our conclusion on a bipartite network translates to that on a two-sided market where two platforms compete.

Proposition 10 *Suppose that the buyer network G is bipartite with the buyer partition (I_1, I_2). For any $h > 0$, there exists $\bar{\varepsilon} > 0$ such that if the externalities are*

ε-close to h-linear for $\varepsilon < \bar{\varepsilon}$, then there exists an SPE (p^*, q^*, σ) in which one seller captures all the buyers. The SPE prices (p^*, q^*) are such that

$$p_i^* = q_i^* = \begin{cases} v^{d_i} - v^0 & \textit{for } i \in I_1, \textit{ and} \\ v^0 - v^{d_i} & \textit{for } i \in I_2. \end{cases}$$

The construction of the equilibrium in Proposition 10 involves the play of the extreme equilibrium in the buyers' subgame that least favors the deviating seller as in Proposition 2. According to Proposition 10, every buyer on one side of the market is taxed whereas those on the other side are subsidized in equilibrium. Such a pricing strategy is in line with a frequent observation in two-sided markets that one side receives a heavy discount. For example, Kaiser and Wright (2006) identify a magazine market in Germany as a two-sided market with readers on one side and advertisers on the other, and find that magazines subsidize their readers while making all profits from their advertisers. Caillaud and Jullien (2003) are the first to offer a theoretical justification of the tax-subsidy pricing scheme in a two-sided market by applying the divide-and-conquer argument to price competition in the market with a single agent on each side.[10]

Another critical observation of Proposition 10 is that the equilibrium pricing is *degree-proportional*: The transfer from or to each buyer i is (approximately) proportional to his degree since $v^{d_i} - v^0 \approx hd_i$ under approximate linearity. Figure 4 illustrates Proposition 10 in a star network with five buyers when the externalities satisfy approximate linearity and

$$v^4 - v^0 \geq 4(v^1 - v^0), \tag{20}$$

so that $I_1 = \{1\}$ and $I_2 = \{2, 3, 4, 5\}$. Buyer 1 at the hub is taxed whereas all the buyers in the periphery are subsidized. We can interpret the subsidies to the peripheral buyers as a protection against the inducement from the other seller. In fact, when (20) holds, it is relatively more difficult for the other seller, say seller A, to induce the hub buyer to switch: When for example all buyers face $q_i = 0$, seller A must pay buyer 1 more than $v^4 - v^0$ to induce him by making $x_1 = A$ dominant (in S^0), whereas he needs to pay just above $4(v^1 - v^0)$ to induce all peripheral buyers by making $x_i = A$ dominant (in S^0). When the inequality (20) is reversed, then buyer

[10]Alternative explanation of the tax-subsidy scheme in two-sided markets is provided by Bolt and Tieman (2008), and Parker and Van Alstyne (2005) among others.

1 now receives a subsidy $v^4 - v^0$, whereas the peripheral buyers are charged $v^1 - v^0$. In this case, hence, the hub buyer needs to be protected as it is relatively easier for the other seller to induce him to switch. As seen in this example, the specification of externalities determines which buyer(s) should be protected with subsidies.

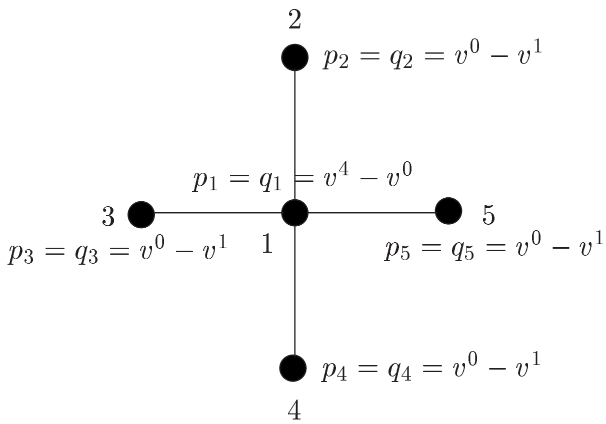

Figure 4: Monopolization through discriminatory pricing on a star network when $v^4 - v^0 \geq 4(v^1 - v^0)$.

When a bipartite network is complete as in the star network above, further characterization of the equilibrium pricing in Proposition 10 is possible. We say that *the marginal externalities are increasing* if

$$v^1 - v^0 \leq v^2 - v^1 \leq \cdots \leq v^D - v^{D-1},$$

and *decreasing* if

$$v^D - v^{D-1} \leq \cdots \leq v^2 - v^1 \leq v^1 - v^0.$$

Under increasing marginal externalities, any buyer in a complete bipartite network is subsidized in equilibrium if and only if his side of the market is larger than the other side. The opposite holds under decreasing marginal externalities.

Corollary 11 *Suppose that the network is complete bipartite with partition (I_1, I_2) such that $n_1 = |I_1| \leq |I_2| = n_2$. For any $h > 0$, there exists $\bar{\varepsilon} > 0$ such that if the externalities are ε-close to h-linear for $\varepsilon < \bar{\varepsilon}$, then then there exists an SPE (p, q, σ) such that*

$$p_i = q_i = \begin{cases} v^{n_2} - v^0 & \textit{for } i \in I_1, \\ v^0 - v^{n_1} & \textit{for } i \in I_2, \end{cases}$$

when the marginal externalities are increasing, and

$$p_i = q_i = \begin{cases} v^0 - v^{n_2} & \textit{for } i \in I_1, \\ v^{n_1} - v^0 & \textit{for } i \in I_2, \end{cases}$$

if the marginal externalities are decreasing.

8 Segmentation on a Bilocular Network

Maintaining the assumption of approximately linear externalities as in the previous section, we now examine the possibility of an equilibrium in which market segmentation takes place. For this, we consider a class of buyer networks that have roughly the opposite property to the bipartite networks in the previous section: In this class of networks, the buyer set is again partitioned into two disjoint subsets, but each buyer has at least as many neighbors in the same subset than in the other subset. Formally, the buyer network is *bilocular* if there exists a two-way partition (I_1, I_2) of the set I of buyers such that for m, $n = 1$, 2, and $m \neq n$,

$$|N_i \cap I_n| \geq |N_i \cap I_m| \text{ for every } i \in I_n, \text{ and}$$
$$|N_i \cap I_n| > |N_i \cap I_m| \text{ for some } i \in I_n.$$

Intuitively, in a bilocular network with partition (I_1, I_2), we can classify buyers in I_1 or I_2 into *core* and *peripheral* buyers: The core buyers are those who have strictly more neighbors in the same set than in the other set, while the peripheral buyers have as many neighbors in the same set as in the other set. One interpretation of a bilocular network is that each one of I_1 and I_2 is a group of traders who trade within their own group more often than outside it. The sellers can then be interpreted as offering platforms to those traders.[11]

A line of four or more buyers is bilocular if I_1 consists of buyers on the left, I_2 consists of buyers on the right, and $|I_1|$, $|I_2| \geq 2$.[12] The buyers on the two ends can be taken as core buyers in this case. The regular network in Figure 3 is also bilocular when we take $I_1 = \{1, 2, 3, 4\}$ and $I_2 = \{5, 6, 7, 8\}$. Buyer 2 and 3 are core buyers for I_1 and buyers 6 and 7 are core buyers for I_2.

Proposition 12 *Suppose that G is bilocular. For any $h > 0$, there exists $\bar{\varepsilon} > 0$ such that if the externalities are ε-close to h-linear for $\varepsilon < \bar{\varepsilon}$, there exists an SPE in*

[11] This interpretation is suggested by Hitoshi Matsushima.

[12] Hence, a bilocular network can be bipartite and vice versa.

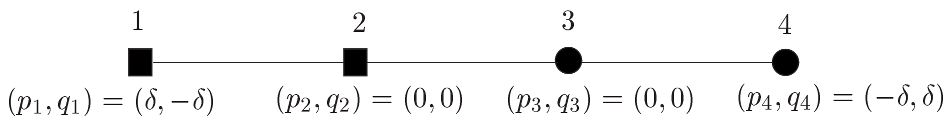

Figure 5: Segmentation on a line network ($\delta = |v^2 + v^1 - 2v^0| > 0$): A captures $I_1 = \{1, 2\}$ and B captures $I_2 = \{3, 4\}$.

which buyers in I_1 choose seller A and buyers in I_2 choose seller B. In this SPE, $p_{i_1} = -q_{i_1} = \delta$ for a single core buyer $i_1 \in I_1$, $p_{i_2} = -q_{i_2} = -\delta$ for a single core buyer $i_2 \in I_2$, and $p_i = q_i = 0$ for all other buyers i, where

$$\delta = \max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right).$$

Note that δ is strictly positive under generic externalities (Lemma 6), small under approximate linearity, and equal to zero under exact linearity. Each seller's equilibrium payoff equals δ, while the sum of their prices over all buyers equals zero. Figure 5 illustrates the equilibrium for a line network of four buyers. As in Proposition 2, any deviation by either seller results in the play of the extreme equilibrium that least favors the deviating seller. Each core buyer who is charged δ will not switch to the other seller since externalities are strictly higher when he consumes the same good as the majority of his neighbors. Intuitively, seller A cannot benefit from any divide-and-conquer strategy since it yields at most

$$\begin{aligned} \sum_{i=1}^{N} \min \left\{ v^{s_i} - v^{d_i - s_i} + q_i^*, \, v^{s_i} \right\} &\leq \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right) + \sum_{i=1}^{N} q_i^* \\ &= \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right), \end{aligned}$$

which is less than or equal to his current payoff δ.[13]

9 Discussion

In this paper, we formulate a model of price competition between two sellers when each one of their goods exhibits local network externalities as represented by a graph-theoretic network of buyers. We show that whether a given price profile is consistent

[13]The proof also shows that attracting a proper subset of buyers is not profitable.

with a subgame perfect equilibrium of the two-stage game depends crucially on the exact specifications of network structure and externalities. In the non-generic case of linear externalities, the marginal cost pricing of both sellers quoting zero to every buyer is consistent with an SPE for any network. Under the generic specification of externalities, however, it is consistent with an SPE if and only if the network is either cyclic or complete. That is, in any other networks, some form of price discrimination is inevitable even if every buyer has exactly symmetric locations in those networks. Given these results, we proceed to the identification of an SPE when the externalities are approximately linear. In a bipartite network which corresponds to a two-sided market, we show that there exists an SPE in which one of the sellers monopolizes the market by charging a positive price to every buyer on one side, and a negative price to every buyer on the other side. The equilibrium prices are approximately proportional to the size of the other side of the market for each buyer. In a bilocular network in which each buyer has more neighbors on his side than on the other side, on the other hand, we show that there exists an equilibrium in which the two sellers segment the market and earn positive profits.

As is well recognized, the essential feature of the market for goods with network externalities is the multiplicity of equilibria. In our context, this corresponds to the multiplicity of equilibria in the buyers' subgame. Note, however, that our impossibility result on marginal-cost pricing holds true regardless of which one of these multiple equilibria may be chosen. On the other hand, our construction of an equilibrium is based on the assumption that following any deviation by either seller, the buyers coordinate on the extreme equilibrium that least favors the deviator. This is a significant departure from the literature which restricts the action profile in the buyers' subgame in one way or the other. While our assumption supports the broadest spectrum of equilibrium in the price competition game, it is not consistent with, for example, the assumption that the buyers choose the Pareto efficient alternative whenever there is one. We think that our exercise is useful as a benchmark given that there is no general consensus on what type of coordination is likely achieved. One related issue concerns what happens when one of the sellers, say seller A, is *focal* as assumed in Jullien (2011). In our terminology, this translates to assuming that the buyers play the A-maximal NE following any price profile. In this case, any monopolization equilibrium identified in this paper is valid with seller A acting as a monopolist. On the other hand, market segmentation is difficult to sustain in equilibrium. Hence, if and how the buyers coordinate their actions have a significant impact on the scope of the equilibrium outcome.

In the present model, the goods of the two sellers are assumed symmetric and incompatible with each other. A natural extension would involve introducing asymmetry or a positive degree of compatibility between them. Technically, introduction of compatibility implies the failure of supermodularity in the buyers' subgame. Endogenous determination of compatibility levels by the sellers is one topic that has received much attention in the literature. For example, Baake and Boom (2001) find in their model of global network externalities that the sellers always choose to offer compatibility in equilibrium. Whether or not the same conclusion holds under local network externalities remains to be seen.

As discussed earlier, the informational assumptions of our model are rather extreme. For example, we assume that each buyer observes the price offers to all other buyers, and that the sellers have perfect knowledge about the buyer network. Relaxing each one of these assumptions yields an interesting model to explore.[14]

Appendix

Proof of Lemma 3. For simplicity, let $k = K$, where K is such that no buyer has a dominant action in S^{k+1} for $k \geq K$. Suppose to the contrary that $1, 2 \in Q_K(p, q)$ and that 1 and 2 are adjacent. Then it must be the case that

$$v^{\alpha_1^K} - p_1 > \max\{v^{\beta_1^K} - q_1, 0\} \quad \text{and} \quad v^{\alpha_2^K} - p_2 > \max\{v^{\beta_2^K} - q_2, 0\}, \tag{21}$$

where for $i = 1$ and 2, recall that

$$\alpha_i^K = \left| N_i \cap \left\{ j : \{A\} = S_j^{K-1}(p, q) \right\} \right| = \left| N_i \cap \left(\cup_{\ell=1}^{K-1} Q_\ell(p, q) \right) \right|$$

is the number of i's neighbors for whom $x_j = A$ is iteratively dominant in round $K - 1$ or earlier, and

$$\beta_i^K = \left| N_i \cap \left\{ j \in I : B \in S_j^{K-1}(p, q) \right\} \right|$$

is the number of i's neighbors for whom $x_j = B$ is not dominated in round $K - 1$ or earlier. (21) can be rewritten as

$$p_1 < \min\left\{ v^{\alpha_1^K} - v^{\beta_1^K} + q_1,\ 0 \right\} \quad \text{and} \quad p_2 < \min\left\{ v^{\alpha_2^K} - v^{\beta_2^K} + q_2,\ 0 \right\}.$$

[14] Pasini *et al.* (2008) study price dispersion in a model of a two-sided market where sellers only know the degree distribution of the buyers.

On the other hand, let p' be such that $p'_i = p_i$ for $i \neq 2$, and

$$p_2 < v^{\alpha_2^K} - \max\{v^{\beta_2^K} - q_2, 0\} < p'_2 < v^{\alpha_2^K+1} - \max\{v^{\beta_2^K-1} - q_2, 0\}.$$

Consider now $Q'_k = Q_k(p', q)$, the set of buyers for whom $x_i = A$ is a dominant action in round k under (p', q). We then have $Q_k(p', q) = Q_k(p, q)$ for $k = 1, \ldots, K-1$ and $Q_K(p', q) = Q_K(p, q) \setminus \{2\}$. Since $1 \in N_2$, this implies that in round $K+1$,

$$\left| N_2 \cap \left\{ j : \{A\} = S_j^K(p', q) \right\} \right| = \alpha_2^K + 1$$

and

$$\left| N_2 \cap \left\{ i \in I : B \in S_j^K(p', q) \right\} \right| = \beta_2^K - 1.$$

Furthermore, by our choice of p'_2,

$$v^{\alpha_2^K+1} - p'_2 > \max\{v^{\beta_2^K-1} - q_2, 0\},$$

which shows that $x_2 = A$ is dominant for buyer 2 in round $K+1$ under (p', q): $Q_{K+1}(p', q) = \{2\}$. Since $p'_2 > p_2$, $\pi_A(p', q, \sigma_B) > \pi_A(p, q, \sigma_B)$, and hence p is not a best response to (q, σ^B). ■

Proof of Lemma 4. Fix any relabeling of buyers $i_1, \ldots, i_N$. Let $s = (s_i)_{i \in I}$ be defined by

$$s_{i_1} = 0 \quad \text{and} \quad s_{i_k} = |N_{i_k} \cap \{i_1, \ldots, i_{k-1}\}| \text{ for } k = 2, \ldots, N.$$

Let $\varepsilon > 0$ be given, and define the price vector $p = (p_i)_{i \in I}$ by

$$p_i = \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\} - \varepsilon. \tag{22}$$

As explained in the text, by offering p, seller A makes $x_{i_1} = A$ a dominant action for buyer i_1, and in any subsequent step, $x_{i_k} = A$ an iteratively dominant action for buyer i_k under (p, q^*). Hence, seller A's payoff under (p, q^*, σ) satisfies

$$\pi_A(p, q^*, \sigma) \geq \sum_{i=1}^{N} \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\} - N\varepsilon.$$

Since $\varepsilon > 0$ and $s \in S$ are arbitrary, if (10) does not hold, then we would have a contradiction

$$\pi_A(p, q^*, \sigma) > \pi_A(p^*, q^*, \sigma).$$

The symmetric argument proves the inequality for seller B's payoff. ■

Proof of Proposition 5. We first show that $(p^*, q^*) = (0, 0)$ is an SPE price. Let σ^A and σ^B be the A-maximal and B-maximal equilibria as defined earlier, and let σ be the buyers' strategy profile such that

$$\sigma(p,q) = \begin{cases} (B,\dots,B) & \text{if } (p,q) = (0,0), \\ \sigma^B(p,q) & \text{if } p \neq 0 \text{ and } q = 0, \\ \sigma^A(p,q) & \text{if } p = 0 \text{ and } q \neq 0. \end{cases}$$

Now consider a deviation from $p^* = 0$ to $p \neq 0$ by seller A. Let $Q_k = Q_k(p, q^*)$ $(k = 1, \dots, K)$ be the set of buyers for whom $x_i = A$ is dominant in round k of the iteration process under (p, q^*) as defined in (4). By the definition of σ^B, a buyer chooses seller A if and only if it is iteratively dominant for him:

$$I_A(\sigma^B(p, q^*)) = \cup_{k=1}^K Q_k.$$

Hence, seller A's payoff under (p, q^*, σ) can be written as:

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^K \sum_{i \in Q_k} p_i. \tag{23}$$

Now recall that

$$\alpha_i^k = \left| N_i \cap \left\{ j : \{A\} = S_j^{k-1} \right\} \right| = \left| N_i \cap \left(\cup_{\ell=1}^{k-1} Q_\ell \right) \right| \quad \text{and} \quad \beta_i^k = \left| N_i \cap \left\{ j : B \in S_j^{k-1} \right\} \right|$$

denote the number of neighbors of buyer i for whom $x_i = A$ is dominant in round $k-1$ or earlier, and $x_i = B$ is undominated in S^{k-1}, respectively. Since $q_i^* = 0$, $x_i = \emptyset$ is dominated by $x_i = B$ in S^0. If follows that

$$N_i \cap \{i : B \in S_{k-1}\} = N_i \setminus \left(\cup_{\ell=1}^{k-1} Q_\ell \right) \quad \Leftarrow \quad \beta_i^k = d_i - \alpha_i^k.$$

If $i \in Q_k$, hence, we should have by (7),

$$v^{\alpha_i^k} - p_i > \max\left\{ v^{\beta_i^k} - q_i^*, 0 \right\} = v^{d_i - \alpha_i^k} \quad \Leftrightarrow \quad p_i < v^{\alpha_i^k} - v^{d_i - \alpha_i^k} = h\left(2\alpha_i^k - d_i \right). \tag{24}$$

Note now that

$$
\begin{aligned}
&\sum_{k=1}^{K} \sum_{i\in Q_k} \alpha_i^k \\
&= \sum_{k=1}^{K} \Big(\#\text{links between } Q_k \text{ and } \cup_{\ell=1}^{k-1} Q_\ell\Big) \\
&\le \#\text{links within } \cup_{k=1}^{K} Q_k \\
&\le \frac{1}{2} \sum_{k=1}^{K} \sum_{i\in Q_k} d_i.
\end{aligned} \tag{25}
$$

Substituting (24) and (25) into (23), we obtain

$$\pi_A(p, q^*, \sigma^B) < h \sum_{k=1}^{K} \sum_{i\in Q_k} \Big(2\alpha_i^k - d_i\Big) \le 0.$$

Therefore, p is not a profitable deviation. The symmetric argument shows that seller B has no profitable deviation q. ■

Proof of Lemma 6. Suppose that s and $d-s$ are *not* permutations of each other. Then (15) implies that either $\sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i}\right) > 0$ or < 0. If the latter holds, then let $i'_k = i_{N-k+1}$ for $k = 1, \ldots, N$ and define $t = (t_i)_{i\in I}$ by setting $t_{i'_k}$ equal to the number of neighbors of i'_k in $\{i'_1, \ldots, i'_{k-1}\}$:

$$t_{i'_1} = 0 \quad \text{and} \quad t_{i'_k} = |N_{i'_k} \cap \{i'_1, \ldots, i'_{k-1}\}| \text{ for } k = 2, \ldots, N. \tag{26}$$

Then we can verify that

$$\sum_{i=1}^{N} \left(v^{t_i} - v^{d_i - t_i}\right) = -\sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i}\right) > 0.$$

Therefore, in order to prove (16), it suffices to show that $d - s$ is not a permutation of s for some s. We will consider the following two cases separately.

1) G is not regular.

Take a pair of buyers i and j such that i is adjacent to j, $d_i = D$ and $d_j < D$, where $D \ge 2$ is the highest degree in G. Take another buyer k that is adjacent to i but not to j. To see that there exists such a buyer, suppose to the contrary that every buyer $\ne j$ that is adjacent to i is also adjacent to j. Then j has at least D neighbors, a contradiction. Let $i_1 = k$, $i_2 = i$ and $i_3 = j$, and define $i_4, \ldots, i_N \notin \{i, j, k\}$ arbitrarily. Then

$$\begin{aligned} &(s_{i_1}, s_{i_2}, s_{i_3}) = (0,1,1)\,, \\ &(d_{i_1} - s_{i_1}, d_{i_2} - s_{i_2}, d_{i_3} - s_{i_3}) = (d_k, D-1, d_j - 1)\,. \end{aligned} \tag{27}$$

If s is not a permutation of $d - s$, then we are done. Suppose then that s is a permutation of $d - s$, and define $i'_1 = k$, $i'_2 = j$, $i'_3 = i$, and $i'_\ell = i_\ell$ for $\ell \geq 4$, and let $t = (t_i)_{i \in I}$ be defined by (26) for these $i'_1, \ldots, i'_N$. Then

$$\begin{aligned} &\left(t_{i'_1}, t_{i'_2}, t_{i'_3}\right) = (0,0,2)\,, \\ &\left(d_{i'_1} - t_{i'_1}, d_{i'_2} - t_{i'_2}, d_{i'_3} - t_{i'_3}\right) = (d_k, d_j, D-2)\,. \end{aligned} \tag{28}$$

Since $i'_\ell = i_\ell$ for $\ell \geq 4$, we have

$$\begin{aligned} &\left|\left\{\ell \geq 4 :\, d_{i_\ell} - s_{i_\ell} = 0\right\}\right| = \left|\left\{\ell \geq 4 :\, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right|, \\ &\left|\left\{\ell \geq 4 :\, s_{i_\ell} = 0\right\}\right| = \left|\left\{\ell \geq 4 :\, t_{i'_\ell} = 0\right\}\right|. \end{aligned} \tag{29}$$

a) $d_j = 1$.

In this case, (27) and (28) imply that

$$\left|\left\{\ell \leq 3 :\, d_{i_\ell} - s_{i_\ell} = 0\right\}\right| = \left|\left\{\ell \leq 3 :\, s_{i_\ell} = 0\right\}\right| = 1.$$

Hence, since $d - s$ is a permutation of s, we must have

$$|\{\ell \geq 4 :\, d_{i_\ell} - s_{i_\ell} = 0\}| = |\{\ell \geq 4 :\, s_{i_\ell} = 0\}|\,.$$

It then follows from (29) that

$$\left|\left\{\ell \geq 4 :\, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| = \left|\left\{\ell \geq 4 :\, t_{i'_\ell} = 0\right\}\right|. \tag{30}$$

However,

$$\left|\left\{\ell \leq 3 :\, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| \leq 1 < 2 = \left|\left\{\ell \leq 3 :\, t_{i'_\ell} = 0\right\}\right|. \tag{31}$$

(30) and (31) together show that $d - t$ cannot be a permutation of t.

b) $d_j \geq 2$.

In this case, we have $D \geq 3$ since $D > d_j \geq 2$, and also

$$\left|\left\{\ell \leq 3 :\, d_{i_\ell} - s_{i_\ell} = 0\right\}\right| = 0 < 1 = \left|\left\{\ell \leq 3 :\, s_{i_\ell} = 0\right\}\right|.$$

Hence, since $d - s$ is a permutation of s,

$$|\{\ell \geq 4 : d_{i_\ell} - s_{i_\ell} = 0\}| = |\{\ell \geq 4 : s_{i_\ell} = 0\}| + 1.$$

It then follows from (29) that

$$\left|\left\{\ell \geq 4 : d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| = \left|\left\{\ell \geq 4 : t_{i'_\ell} = 0\right\}\right| + 1. \tag{32}$$

However, (27) and (28) imply that

$$\left|\left\{\ell \leq 3 : d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| = \left|\left\{\ell \leq 3 : t_{i'_\ell} = 0\right\}\right| - 2 \tag{33}$$

(32) and (33) together imply that $d - t$ is not a permutation of t.

2) G is r-regular with $2 < r < N - 1$.

Since G is connected and not complete, we can take a pair of buyers i_1 and i_2 such that i_1 and i_2 are adjacent, and take another buyer i_3 who is adjacent to i_2 but not to i_1. To see that this is possible, suppose to the contrary that for any pair of adjacent buyers i and j, any buyer $k \neq i$ adjacent to j is also adjacent to i. We then show that G must be complete. Take any pair of buyers i and j. Since G is connected, there is a path $k_1 = i \to k_2 \to \cdots \to k_{m-1} \to k_m = j$. Since k_2 is adjacent to $i = k_1$ and k_3 is adjacent to k_2, k_3 is adjacent to i as well by the above. Now since k_4 is adjacent to k_3, it is also adjacent to i. Proceeding the same way, we conclude that $j = k_m$ is adjacent to $i = k_1$, implying that G is complete.

We now label buyers other than $\{i_1, i_2, i_3\}$ as $i_4, \ldots, i_N$ in an arbitrary manner. For our choice of i_1, i_2 and i_3, we have

$$\begin{aligned}
&(s_{i_1}, s_{i_2}, s_{i_3}) = (0, 1, 1)\,, \\
&(d_{i_1} - s_{i_1}, d_{i_2} - s_{i_2}, d_{i_3} - s_{i_3}) = (r, r-1, r-1)\,.
\end{aligned}$$

If $d - s$ is a not permutation of s, then we are done. Suppose then that $d - s$ is a permutation of s. We then must have

$$\left|\{\ell : s_{i_\ell} = 0\}\right| = \left|\{\ell : d_{i_\ell} - s_{i_\ell} = 0\}\right|. \tag{34}$$

Let $i'_1 = i_1$, $i'_2 = i_3$, $i'_3 = i_2$ and $i'_\ell = i_\ell$ for $\ell \geq 4$, and let $t = (t_i)_{i \in I}$ be defined by (26) for these $i'_1, \ldots, i'_N$. Note that

$$\begin{aligned}
&\left(t_{i'_1}, t_{i'_2}, t_{i'_3}\right) = (0, 0, 2)\,, \\
&\left(d_{i'_1} - t_{i'_1}, d_{i'_2} - t_{i'_2}, d_{i'_3} - t_{i'_3}\right) = (r, r, r-2)\,.
\end{aligned}$$

Since $r > 2$, if (34) holds, then the same argument as in the non-regular case shows that

$$\left|\left\{\ell : t_{i'_\ell} = 0\right\}\right| \neq \left|\left\{\ell : d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right|,$$

implying that $d - t$ is not a permutation of t. ■

Proof of Lemma 7. We first show that if (p^*, q^*, σ) is an SPE, then

$$\pi_A(p^*, q^*, \sigma) > \sum_{i=1}^{N} \min\{q_i^*,\, v^0\} \quad \text{and} \quad \pi_B(p^*, q^*, \sigma) > \sum_{i=1}^{N} \min\{p_i^*,\, v^0\}. \qquad (35)$$

By Lemma 4, for any $s \in S$, seller A's payoff under (p^*, q^*) satisfies

$$\pi_A(p^*, q^*, \sigma) \geq \sum_{i=1}^{N} \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\}.$$

Rearranging, we get for any $s \in S$,

$$\begin{aligned}\pi_A(p^*, q^*, \sigma) &\geq \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i}\right) + \sum_i \min\{q_i^*,\, v^{d_i - s_i}\} \\ &\geq \sum_{i=1}^{N} (v^{s_i} - v^{d_i - s_i}) + \sum_{i=1}^{N} \min\{q_i^*,\, v^0\}.\end{aligned}$$

When G is neither cyclic or complete, there exists by Lemma 6 an $s \in S$ such that the first term on the right-hand side is > 0. Hence, the first inequality in (35) must hold. The proof for the second inequality is similar.

a) If $\min_i q_i^* \geq 0$, then $\pi_A(p^*, q^*, \sigma) = 0 \leq \sum_i \min\{q_i^*, v^0\}$, contradicting (35).

b) If $\max_i q_i^* \leq v^0$, then $\pi_A(p^*, q^*, \sigma) \leq \sum_i q_i^* = \sum_i \min\{q_i^*, v^0\}$, contradicting (35).

c) The inequality $\max_i q_i^* > v^0$ follows from (b) above since $I_B(p^*, q^*, \sigma) = I$ implies $\pi_A(p^*, q^*, \sigma) = 0$ and $0 \leq \pi_B(p^*, q^*, \sigma) = \sum_i q_i^*$. If $v^{d_i} - q_i^* < v^0$ for some i, then any p such that $p_i = v^0 - \varepsilon$ and $p_{-i} = 0$ for $0 < \varepsilon < q_i^* - v^{d_i} + v^0$ would induce buyer i to switch to A and hence is a profitable deviation for seller A. To see that $v^D > 2v^0$, note first that $\min_i (v^{d_i} - q_i^*) \geq v^0$ in particular implies that $\max_i q_i^* \leq v^D - v^0$. Hence, if $v^D \leq 2v^0$, we have a contradiction to the first statement since $\max_i q_i^* \leq v^D - v^0 \leq v^0$. ■

Proof of Proposition 8. Suppose that G is neither cyclic or complete, and suppose that seller B attracts all the buyers in an SPE (p^*, q^*, σ) such that $q_1^* = \cdots = q_N^*$. Then since $\pi_A(p^*, q^*, \sigma) = 0$, Lemma 7(1) implies that $q_1^* = \cdots = q_N^* = \min_i q_i^* < 0$. Then, however, $\pi_B(p^*, q^*, \sigma) < 0$, a contradiction. ■

Proof of Proposition 9 It suffices to show that in each class of networks, (p^*, q^*, σ) is an SPE when $(p^*, q^*) = (0, 0)$ and the buyers' strategy profile σ is such that

$$\sigma(p,q) = \begin{cases} (B,\dots,B) & \text{if } (p,q) = (p^*,q^*), \\ \sigma^A(p,q) & \text{if } p = p^* \text{ and } q \neq q^*, \\ \sigma^B(p,q) & \text{if } p \neq p^* \text{ and } q = q^*. \end{cases}$$

In other words, all buyers choose B under $(p^*, q^*) = (0, 0)$, and when one of the firms deviates to a non-zero price vector, the buyers coordinate on the extreme NE which least favors the deviating seller. In what follows, we show that seller A has no profitable deviation. A symmetric argument shows that seller B has no profitable deviation.

1) G is a cycle.

Suppose that seller A deviates to $p \neq p^*$. Let $Q_k = Q_k(p, q^*)$ be the set of buyers for whom $x_i = A$ is dominant in round k under (p, q^*) as defined in (4). By the definition of σ^B, buyer i chooses A if and only if $x_i = A$ is iteratively dominant:

$$I_A(\sigma^B(p, q^*)) = \cup_{k=1}^K Q_k.$$

Since G is cyclic, $d_i = |N_i| = 2$, where N_i is the set of neighbors of i. Recall that

$$\alpha_i^k = \left|N_i \cap \left\{j : \{A\} = S_j^{k-1}\right\}\right| = \left|N_i \cap \left(\cup_{\ell=1}^{k-1} Q_\ell\right)\right| \quad \text{and} \quad \beta_i^k = \left|N_i \cap \left\{j : B \in S_j^{k-1}\right\}\right|$$

denote the number of neighbors of i for whom $x_i = A$ is dominant in round $k-1$ or earlier, and $x_i = B$ is undominated in S^{k-1}, respectively. Since $q_i^* = 0$, $x_i = \emptyset$ is dominated by $x_i = B$ in S^0 for any buyer i. It follows that $B \in S^{k-1}$ if and only if $\{A\} \subsetneq S_i^{k-1}$, and hence that

$$N_i \cap \{i : B \in S_{k-1}\} = N_i \cap \left(\cup_{\ell=1}^{k-1} Q_\ell\right) \quad \Leftarrow \quad \beta_i^k = 2 - \alpha_i^k.$$

Suppose now that $i \in Q_k$. Then we have by (7),

$$v^{\alpha_i^k} - p_i > v^{2-\alpha_i^k} \quad \Leftrightarrow \quad p_i < v^{\alpha_i^k} - v^{2-\alpha_i^k}.$$

In particular, $i \in Q_1$ in round 1 if $p_i < v^0 - v^2$, and $i \in Q_k$ in round $k > 1$ either if (i) $p_i < 0$ and exactly one of his two neighbors has already chosen A ($\alpha_i^k = 1$), or (ii) $p_i < v^2 - v^0$ and both his neighbors have already chosen A ($\alpha_i^k = 2$). Note in particular that if buyer i finds $x_i = A$ dominant when neither of his neighbors have already chosen A, then $i \in Q_1$.

Seller A's payoff under (p, q^*, σ) hence satisfies

$$\begin{aligned}\pi_A(p, q^*, \sigma) &= \sum_{k=1}^{K} \sum_{i \in Q_k} p_i \\ &< |Q_1|(v^0 - v^2) + (v^2 - v^0) \sum_{k=2}^{K} \left| \{ i \in I \setminus \left(\cup_{\ell=1}^{k-1} Q_\ell \right) \,:\, \alpha_i^k = 2 \} \right| .\end{aligned}$$

Since no buyer finds A dominant in round $k \geq 2$ if neither of his neighbors has already chosen A, the number of components (*i.e*, connected clusters of buyers) in $\cup_{\ell=1}^{k-1} Q_\ell$ is less than or equal to that in Q_1 for any k. It follows that

$$\sum_{k=2}^{K} |\{ i \in I \setminus \left(\cup_{\ell=1}^{k-1} Q_\ell \right) \,:\, \alpha_i^k = 2 \}| \leq |Q_1|.$$

We can therefore conclude that $\pi_A(p, q^*, \sigma) \leq 0$ and hence that p is not a profitable deviation.

2) G is complete.

Define $Q_k = Q_k(p, q^*)$ ($k = 1, \ldots, K$) as above. Denote by α^k the number α^k of buyers who have chosen A in rounds $1, \ldots, k-1$:

$$\alpha^k = \sum_{\ell=1}^{k-1} |Q_\ell| \,.$$

Since G is complete, for any buyer i, the number α_i^k of i's neighbors who have chosen A equals α^k. Furthermore, by Proposition 3, we only need consider p such that each Q_k contains a single buyer. (If Q_k contains two or more buyers, then since G is complete, those buyers are adjacent.) Hence, without loss of generality, $Q_k = \{k\}$ for each $k = 1, \ldots, N$. For $k = 1, \ldots, K$, we also have

$$p_k < v^{\alpha_k} - v^{N-1-\alpha_k}.$$

Seller A's payoff under (p, q^*, σ) hence satisfies

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^{N} \sum_{k=1}^{N} p_k < \sum_{k=1}^{K} \left(v^{\alpha^k} - v^{N-1-\alpha^k} \right). \tag{36}$$

It is then straightforward to verify that the right-hand side equals zero. Hence, seller A has no profitable deviation. ■

Proof of Proposition 10. We will construct an SPE (p^*, q^*, σ) in which (p^*, q^*) is as given in the proposition, and

$$\sigma(p,q) = \begin{cases} \sigma^B(p,q) & \text{if } q = q^*, \\ \sigma^A(p,q) & \text{otherwise.} \end{cases}$$

Since $\sigma(p^*, q^*)$ is B-maximal and $p^* = q^*$, no buyer chooses A. Furthermore, no buyer chooses $\emptyset$ since $x = (B, \dots, B)$ yields non-negative payoffs to all buyers. It follows that seller B monopolizes the market following (p^*, q^*).

Consider any deviation $p \neq p^*$ by seller A. Since it induces the B-maximal NE $\sigma^B(p, q^*)$, the set of buyers who choose A equals $I_A(\sigma(p, q^*)) = \cup_{k=1}^{K} Q_k$, where $Q_k = Q_k(p, q^*)$ is the set of buyers i for whom $x_i = A$ is dominant in round k under (p, q^*) as defined in (4). For any $J \subset I$, define

$$N_J = \cup_{j \in J} N_j$$

to be the collection of neighbors of buyers in J. Let also $Q_0 = \emptyset$ and $N_\emptyset = \emptyset$.

Step 1. If $i \in Q_k$ for some $k \geq 2$, then $N_i \cap Z_{k-1} \neq \emptyset$.

If $N_i \cap Z_{k-1} = \emptyset$, then for any $j \in N_i$ such that $B \in S_j^{k-2}$, $x_j = B$ is undominated in S^{k-2} by the definition of Z_{k-1}. It follows that $N_i \cap Q_{k-1} = \emptyset$,

$$\alpha_i^{k-1} = \left| N_i \cap \cup_{\ell=0}^{k-2} Q_\ell \right| = \left| N_i \cap \cup_{\ell=0}^{k-1} Q_\ell \right| = \alpha_i^k,$$

and

$$\beta_i^{k-1} = \left| N_i \cap \left\{ j : B \in S_j^{k-2} \right\} \right| = \left| N_i \cap \left\{ j : B \in S_j^{k-1} \right\} \right| = \beta_i^k.$$

Hence, $x_i = A$ cannot become dominant in S^{k-1} when it is not dominant in S^{k-2}. In other words, $i \notin Q_k$.

Step 2. Suppose that $i \in Q_k$ for some $k \geq 2$.

a) If $i \in I_1$, then $N_i \cap Q_{k-1} \neq \emptyset$.

Take any $j \in N_i \cap Z_{k-1}$, which is $\neq \emptyset$ by Step 1.

For any $j \in I_2$, $x_j = \emptyset$ is dominated by $x_j = B$ in S^0 since $q_j^* > 0$. Hence, $S_j^\ell \subset \{A, B\}$ for $\ell \geq 1$. The fact that $j \in Z_{k-1}$ then implies that $x_j = A$ is dominant in S^{k-2}: $j \in Q_{k-1}$.

b) If $i \in I_2$, then there exists $N_i \cap \left(Q_{k-1} \cup \left(N_{Q_{k-2}} \setminus \cup_{\ell=1}^{k-2} Q_\ell\right)\right) \neq \emptyset$.

Take any $j \in N_i \cap Z_{k-1}$. Since $x_j = B$ is dominated in S^{k-2}, we have

$$v^{\beta_j^{k-1}} - q_j^* < \max\left\{v^{\alpha_j^{k-1}} - p_j,\, 0\right\}.$$

If $v^{\alpha_j^{k-1}} - p_j > 0$, then $x_j = A$ is dominant in S^{k-2} so that $j \in Q_{k-1}$. If $v^{\alpha_j^{k-1}} - p_j \leq 0$, then $v^{\beta_j^{k-1}} - q_j^* < 0$ implies that $\beta_j^{k-1} < d_j$ since $q_j^* = v^{d_j} - v^0$. On the other hand, since $B \in S_j^{k-2}$, $x_j = B$ is undominated in S^{k-3}. Hence, we have $j \notin \cup_{\ell=1}^{k-2} Q_\ell$, and also

$$v^{\beta_j^{k-2}} - q_j^* \geq \max\left\{v^{\alpha_j^{k-2}} - p_j,\, 0\right\} \geq 0,$$

which shows that $\beta_j^{k-2} = d_j$. It follows that for some neighbor $m \in I_2$ of j, $B \in S_m^{k-3}$ but $B \notin S_m^{k-2}$. Since $x_m = B$ can be dominated only by $x_m = A$ for $m \in I_2$, this implies that $m \in Q_{k-2}$. Therefore, $j \in N_{Q_{k-2}} \setminus \left(\cup_{\ell=1}^{k-2} Q_\ell\right)$.

Step 3. If $i \in I_1$, then $\beta_i^k = d_i - \alpha_i^k$.

If $i \in I_1$, $N_i \subset I_2$. Since $q_j^* < 0$ for any $j \in I_2$, $x_j = \emptyset$ is dominated by $x_j = B$ in S^0 for any such j. It follows that $B \in S_j^{k-1}$ if and only if $B \in S_j^{k-2}$ and $x_j = B$ is not dominated by $x_j = A$ in S^{k-2}. Hence, we have by induction,

$$\begin{aligned} N_i \cap \left\{j : B \in S_j^{k-1}\right\} &= N_i \cap \left\{j : B \in S_j^{k-2} \text{ and } j \notin Q_{k-1}\right\} \\ &= N_i \cap \left\{j : B \in S_j^{k-3} \text{ and } j \notin Q_{k-2} \cup Q_{k-1}\right\} \\ &= \cdots \\ &= N_i \setminus \cup_{\ell=1}^{k-1} Q_\ell. \end{aligned}$$

This shows that $\beta_i^k = d_i - \alpha_i^k$.

Step 4. If $\beta_i^k \geq 1$ for some $i \in I_2 \cap Q_k$ and $k \geq 1$, then $\pi_A(p, q^*) < 0$.

Since $\alpha_i^1 = 0$ and $\beta_i^1 = d_i$, $v^{ap_i^1} - v^{\beta_i^1} + q_i^* = 0$ for $i \in I_1 \cap Q_1$. Hence, $i \in I_1 \cap Q_1$ implies that $p_i < 0$ by (7). On the other hand, since $\beta_i^k = d_i - \alpha_i^k$ by Step 3 and $\alpha_i^k \geq 1$ for $k \geq 2$ by Step 2(a), we have $\beta_i^k < d_i$ for $i \in I_1 \cap Q_k$ for $k \geq 2$. Hence, under approximate linearity,

$$v^{ap_i^k} - v^{\beta_i^k} + q_i^* = v^{ap_i^k} - v^{\beta_i^k} + v^{d_i} - v^0 \approx v^{\alpha_i^k} + d_i - \beta_i^k > v^{\alpha_i^k}.$$

By (7), then $i \in I_1 \cap Q_k$ for $k \geq 2$ implies that $p_i < v^{\alpha_i^k}$. It follows that seller A's payoff π_A under (p, q^*) satisfies

$$\begin{aligned}
&\pi_A(p,q^*) \\
&< \sum_{k=1}^{K} \sum_{i \in Q_k} p_i \\
&= \sum_{k=2}^{K} \sum_{i \in I_1 \cap Q_k} v^{\alpha_i^k} + \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} \left(v^{\alpha_i^k} - v^{\beta_i^k} + v^0 - v^{d_i} \right) \\
&= \sum_{i \in I_2 \cap Q_1} v^0 + \sum_{k=2}^{K} \sum_{i \in Q_k} v^{\alpha_i^k} - \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} \left(v^{d_i} - v^0 \right) - \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} v^{\beta_i^k}.
\end{aligned} \tag{37}$$

We will show that $\pi_A(p, q^*) \leq -h$ when the externalities are h-linear and $\beta_i^k \geq 1$ for some $i \in I_2 \cap Q_k$ and $k \geq 1$. This will prove the statement when the externalities are ε-close to h-linear for $\varepsilon > 0$ sufficiently small since $\pi_A(p, q^*)$ is continuous in ε. Under h-linearity, we can rewrite (37) as:

$$\pi_A(p,q^*) < h \sum_{k=2}^{K} \sum_{i \in Q_k} \alpha_i^k - h \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} d_i - h \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} \beta_i^k.$$

Note now that

$$\sum_{i \in Q_k} \alpha_i^k = \sum_{i \in Q_k} \left| N_i \cap \left(\cup_{\ell=1}^{k-1} Q_\ell \right) \right| = \#\text{links between } Q_k \text{ and } \cup_{\ell=1}^{k-1} Q_\ell,$$

and hence that

$$\begin{aligned}
\sum_{k=2}^{K} \sum_{i \in Q_k} \alpha_i^k &= \#\text{links within } \cup_{\ell=1}^{K} Q_\ell \\
&= \#\text{links from } I_2 \cap \left(\cup_{\ell=1}^{K} Q_\ell \right) \text{ to } I_1 \cap \left(\cup_{\ell=1}^{K} Q_\ell \right) \\
&\leq \#\text{links from } I_2 \cap \left(\cup_{\ell=1}^{K} Q_\ell \right) \text{ to } I_1 \\
&= \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} d_i.
\end{aligned}$$

It follows that if $\beta_i^k \geq 1$ for some $i \in I_2 \cap Q_k$ and $k \geq 1$,

$$\pi_A(p,q^*) < -h \sum_{k=1}^{K} \sum_{i \in I_2 \cap Q_k} \beta_i^k \leq -h.$$

Step 5. If $N_i \setminus \cup_{\ell=0}^{k-1} Q_\ell \neq \emptyset$ and $i \in Q_k$ for some $k \geq 1$ and $i \in I_2$, then $\pi_A(p, q^*) < 0$.

If $B \in S_j^{k-1}$ for some $j \in N_i \setminus \cup_{\ell=0}^{k-1} Q_\ell \neq \emptyset$, then

$$\beta_i^k = \left| N_i \cap \left\{ j : B \in S_j^{k-1} \right\} \right| \geq 1.$$

Suppose then that $B \notin S_j^{k-1}$ for every $j \in N_i \setminus \cup_{\ell=0}^{k-1} Q_\ell \neq \emptyset$, and take any such j. Since $q_j^* = v^{d_j} - v^0$, $B \notin S_j^{k-1}$ implies that there exists $m \in N_j \subset I_2$ such that $B \notin S_m^{k-2}$. Since $x_m = \emptyset$ is dominated by $x_m = B$ for any such m, $B \notin S_m^{k-2}$ implies $S_m^{k-2} = \{A\}$, or equivalently, $m \in \cup_{\ell=0}^{k-2} Q_\ell$. It follows that $N_j \cap \cup_{\ell=0}^{k-2} Q_\ell \neq \emptyset$. Take the smallest $k^* \leq k-2$ such that $N_j \cap Q_{k^*} \neq \emptyset$ and $i^* \in N_j \cap Q_{k^*}$. Then $B \in S_j^{k^*-1}$ and hence $\beta_{i^*}^{k^*} \geq 1$.

Step 6. If $Q_k \cap I_1 \neq \emptyset$ for some $k \geq 2$, then $\pi_A(p, q^*) < 0$.

Let $i \in Q_k \cap I_1$. We can take $j \in N_i \cap Q_{k-1} \neq \emptyset$ by Step 2(a). Since $i \notin \cup_{\ell=0}^{k-2} Q_\ell$, we have $N_j \setminus \cup_{\ell=0}^{k-2} Q_\ell \neq \emptyset$. It then follows from Step 3 that $\pi_A(p, q^*) < 0$.

Step 7. If $Q_k \cap I_2 \neq \emptyset$ for some $k \geq 3$, then $\pi_A(p, q^*) < 0$.

Let $i \in Q_k \cap I_2$. We can take $j \in N_i \cap \left(Q_{k-1} \cup \left(N_{Q_{k-2}} \setminus \cup_{\ell=1}^{k-2} Q_\ell \right) \right) \neq \emptyset$ by Step 2(b). If $j \in Q_{k-1}$, then $\pi_A(p, q^*) < 0$ by Step 6 since $j \in I_1$. If $j \in N_{Q_{k-2}} \setminus \cup_{\ell=1}^{k-2} Q_\ell$, then there exists $m \in I_2$ such that $m \in N_j \cap Q_{k-2}$. Since $j \in N_m \setminus \cup_{\ell=1}^{k-2} Q_\ell$, we have $\pi_A(p, q^*) < 0$ by Step 5.

Step 8. If $Q_1 \subset I_1$, $Q_2 \subset I_2$, and $Q_k = \emptyset$ for $k \geq 3$, then $\pi_A(p, q^*) \leq 0$.

First, if $N_i \setminus Q_1 \neq \emptyset$ for some $i \in Q_2$, then $\pi_A(p, q^*) < 0$ by Step 5. Suppose then that $N_i \subset Q_1$ for every $i \in Q_2$. In this case, $\alpha_i^2 = d_i$ and $\beta_i^2 = 0$ for every $i \in Q_2$, and hence

$$\pi_A(p, q^*) < \sum_{i \in Q_1} \max \left\{ v^0 - v^{d_i} + q_i^*, 0 \right\} + \sum_{i \in Q_2} \max \left\{ v^{d_i} - v^0 + q_i^*, 0 \right\} = 0.$$

Proof of Proposition 12. Let

$$\delta = \max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right).$$

When the externalities are ε-close to h-linear,

$$\begin{aligned} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right) &= \sum_{i=1}^{N} \left\{ (v^{s_i} - s_i h) - \left(v^{d_i - s_i} - (d_i - s_i) h \right) - h \left((d_i - s_i) - s_i \right) \right\} \\ &< 2N\varepsilon, \end{aligned}$$

and hence

$$\delta < 2N\varepsilon. \tag{38}$$

Let (I_1, I_2) be the partition of the buyer set I, and let $i_A \in I_1$ and $i_B \in I_2$ be the core buyers of the respective sets:

$$|N_{i_A} \cap I_1| > |N_{i_A} \cap I_2| \text{ and } |N_{i_B} \cap I_2| > |N_{i_B} \cap I_1|.$$

We specify (p^*, q^*, σ) as follows:

$$(p_i^*, q_i^*) = \begin{cases} (\delta, -\delta) & \text{if } i = i_A, \\ (-\delta, \delta) & \text{if } i = i_B, \\ (0, 0) & \text{otherwise,} \end{cases}$$

and

$$\sigma(p, q) = \begin{cases} (\underbrace{A, \ldots, A}_{I_1}, \underbrace{B, \ldots, B}_{I_2}) & \text{if } (p, q) = (p^*, q^*), \\ \sigma^B(p, q) & \text{if } p \neq p^*, \\ \sigma^A(p, q) & \text{if } p = p^* \text{ and } q \neq q^*. \end{cases}$$

Note that $\pi_A(p^*, q^*, \sigma) = \pi_B(p^*, q^*, \sigma) = \delta$.

We first show that the buyers' action profile following (p^*, q^*) is a NE. If $i \in I_1 \setminus \{i_A\}$, then $x_i = A$ is a best response since

$$v^{|N_i \cap I_1|} - p_i = v^{|N_i \cap I_1|} \geq v^{|N_i \cap I_2|} = v^{|N_i \cap I_2|} - q_i.$$

If $i = i_A$, then $|N_i \cap I_1| > |N_i \cap I_2|$ so that

$$\begin{aligned} & v^{|N_i \cap I_1|} - v^{|N_i \cap I_2|} \\ & = \left(v^{|N_i \cap I_1|} - h|N_i \cap I_1|\right) - \left(v^{|N_i \cap I_2|} - h|N_i \cap I_2|\right) + h\left\{|N_i \cap I_1| - |N_i \cap I_2|\right\} \\ & \geq h - 2\varepsilon. \end{aligned}$$

Hence, if we take

$$\bar{\varepsilon} = \frac{h}{2(2N+1)}, \tag{39}$$

then for any $\varepsilon < \bar{\varepsilon}$, (38) implies that

$$v^{|N_i \cap I_1|} - p_i = v^{|N_i \cap I_1|} - \delta > v^{|N_i \cap I_2|} + \delta = v^{|N_i \cap I_2|} - q_i.$$

The symmetric argument shows that $x_i = B$ is a best response for each $i \in I_2$ following (p^*, q^*).

We will next show that seller A has no profitable deviation. Let p be any deviation by seller A, and $Q_k = Q_k(p, q^*)$ be the set of buyers i for whom $x_i = A$ is a dominant action in round k under (p, q^*) as defined in (4). Since the buyers play the B-maximal NE $\sigma^B(p, q^*)$, buyer i choose A if and only if $i \in \cup_{k=1}^K Q_k$. By Lemma 3, we may assume that no buyers in Q_k are adjacent.

Suppose that $i \in Q_k$. For any neighbor $j \in N_i$ of i, we have

$$j \notin \cup_{\ell=1}^{k-1} Q_\ell \quad \Rightarrow \quad B \in S_j^{k-1}. \tag{40}$$

We can see that (40) holds as follows: First, take $j \neq i_B$. Since then $q_j^* \leq 0$, $x_j = B$ is not dominated by $x_j = \emptyset$. Hence, if $x_j = A$ is not dominant in $S^{\ell-1}$ for $\ell = 1, \ldots, k$ (*i.e.*, $j \notin \cup_{\ell=1}^{k-1} Q_\ell$), then $B \in S_j^{k-1}$. On the other hand, if $j = i_B$, then $q_j^* = \delta < 2N\varepsilon < h = v^1$ under approximate linearity. Furthermore, $i \in Q_k$ implies that $\{A\} \neq S_i^{k-1}$. Since $i \neq i_B$, we have $B \in S_i^{k-1}$ by the preceding argument. It follows that $x_j = B$ is not dominated by $x_j = \emptyset$ in $S^{\ell-1}$ for $\ell = 1, \ldots, k$ since

$$v^{\beta_j^\ell} - q_i^* \geq v^1 - \delta > 0,$$

where $\beta_j^k = \left|N_j \cap \left\{m : B \in S_m^{k-1}\right\}\right| \geq 1$. Hence, if $x_j = A$ is not dominant in $S^{\ell-1}$ for $\ell = 1, \ldots, k$ (*i.e.*, $j \notin \cup_{\ell=1}^{k-1} Q_\ell$), then $B \in S_j^{k-1}$.

Recalling that

$$\alpha_i^k = \left|N_i \cap \left(\cup_{\kappa=1}^{k-1} Q_\kappa\right)\right|$$

equals the number of i's neighbors for whom A is dominant prior to round k, we conclude from (40) that $\beta_i^k = d_i - \alpha_i^k$. Hence, (7) shows that if $i \in Q_k$, then p_i satisfies

$$p_i < \min\left\{v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*,\, v^{\alpha_i^k}\right\} \leq v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*,$$

which in turn implies that seller A's payoff π_A under (p, q^*) satisfies

$$\begin{aligned}
\pi_A(p, q^*, \sigma) &= \sum_{k=1}^K \sum_{i \in Q_k} p_i \\
&< \sum_{k=1}^K \sum_{i \in Q_k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*\right) \qquad (41) \\
&\leq \sum_{k=1}^K \sum_{i \in Q_k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k}\right) + \delta.
\end{aligned}$$

We will show that $\pi_A(p, q^*) \leq 0$ for any p by considering the following two cases separately.

Suppose first that $\cup_{k=1}^K Q_k \subsetneq I$. Since the right-hand side of (41) is continuous in ε, if we show that it is less then $-h$ under exact linearity, then $\pi_A(p, q^*) < 0$ holds under approximate linearity. Under exact linearity, (41) becomes

$$\begin{aligned}\pi_A(p, q^*, \sigma) &< \sum_{k=1}^K \sum_{i \in Q_k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} \right) + \delta \\ &= h \sum_{k=1}^K \sum_{i \in Q_k} \left(2\alpha_i^k - d_i \right).\end{aligned}$$

Note that

$$\sum_{k=1}^K \sum_{i \in Q_k} \alpha_i^k = \#\text{links within } \cup_{k=1}^K Q_k,$$

and that

$$\begin{aligned}\sum_{k=1}^K \sum_{i \in Q_k} d_i = 2 \times (&\#\text{links within } \cup_{k=1}^K Q_k) \\ &+ \#\text{links from } \cup_{k=1}^K Q_k \text{ to } I \setminus \cup_{k=1}^K Q_k.\end{aligned}$$

It follows that

$$\pi_A(p, q^*, \sigma) < (-h) \times \#\text{links from } \cup_{k=1}^K Q_k \text{ to } I \setminus \cup_{k=1}^K Q_k \leq -h,$$

where the inequality follows from the fact that $\cup_{k=1}^K Q_k \subsetneq I$.

Suppose next that $\cup_{k=1}^K Q_k = I$. In this case, $\sum_{k=1}^K \sum_{i \in Q_k} q_i^* = 0$ by definition. Hence,

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^K \sum_{i \in Q_k} p_i \leq \sum_{k=1}^K \sum_{i \in Q_k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* \right) \leq \delta = \pi_A(p^*, q^*, \sigma),$$

where the inequality follows from the definition of δ. ■

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